LEIPZIG UNIVERSITY

BACHELOR THESIS

Energy identities of the Klein-Gordon equation's stress-energy-momentum tensor in a Lorentzian spacetime

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Declaration of Authorship

I, Kai Uslar, hereby declare that the contents of this thesis, titled "Energy identities of the Klein-Gordon equation's stress-energy-momentum tensor in a Lorentzian spacetime," is my own work, and that all sources of information have been appropriately cited.

Furthermore, AI software, such as *ChatGPT* and *DeepSeek*, was used only to improve readability and professionalism. All AI-assisted content was carefully reviewed, fact-checked, and rewritten when necessary.

Abstract

This thesis establishes a geometric framework for deriving energy conservation laws in curved spacetime. By combining the stress-energy-momentum tensor of the Klein-Gordon field with the divergence theorem on Lorentzian manifolds, we prove that spacetime symmetries, encoded by Killing vector fields, yield rigorous conservation laws. The results are shown to extend to a broad class of hyperbolic partial differential equations and remain valid under weakened regularity assumptions, demonstrating the universality of this geometric approach to energy conservation.

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Contents

Chapter 1

Introduction

In the flat spacetime of Newtonian mechanics and Special Relativity, the principle of energy conservation is a straightforward foundation of physics, grounded in the absolute nature of time and space.

However, General Relativity revolutionizes this picture by describing spacetime itself as a dynamic, curved entity. In this framework, the familiar global conservation of energy is no longer guaranteed. This raises fundamental questions: How can we meaningfully define conserved quantities like energy in a general curved spacetime? What mathematical structures allow us to recover conservation laws?

This thesis addresses these questions by developing a general framework for deriving conservation laws from hyperbolic field equations on Lorentzian manifolds. While we use the Klein-Gordon equation as our primary example, the core results depend only on the geometric structure of the stress-energy-momentum tensor and the existence of spacetime symmetries, making them applicable to a wide class of physical theories.

1.1 Geometry and Symmetry in Physics

The profound connection between symmetries and conservation laws was rigorously established in the early 20th century by Emmy Noether. [Noe71] Her first theorem states that every differentiable symmetry of a physical system's action corresponds to a conservation law.

In the context of field theory in curved spacetime, this theorem provides the crucial link between geometry and physics:

- The spacetime is modeled as a Lorentzian manifold
- Symmetries are encoded by Killing vector fields
- The conserved quantities arise from the stress-energy-momentum tensor.

Our work uses this mathematical framework to show how local symmetries of spacetime give rise to global conservation laws, even when the spacetime geometry is non-trivial.

1.1.1 The Model System: Klein-Gordon Equation

The Klein-Gordon equation serves as an ideal model for demonstrating this framework. As the simplest relativistic field equation, it provides a mathematically transparent setting while containing all essential features:

- It demonstrates the complete methodology: from field equation to global conservation law
- It clearly shows how spacetime symmetries (Killing vectors) yield conserved quantities
- It illustrates the crucial role of the divergence theorem in Lorentzian geometry.

Crucially, our main mathematical result—the energy identity—depends only on the hyperbolic nature of the PDE and the existence of a divergence-free stress-energy tensor, not on the specific form of the Klein-Gordon equation. This makes the framework broadly applicable to other field theories including electromagnetism and Dirac fields.

1.2 Thesis Roadmap

The thesis is structured as follows:

- **Chapter 2** establishes the mathematical background in differential geometry and partial differential equations necessary for our analysis.
- **Chapter 3** presents and proves the divergence theorem in a Lorentzian setting, which serves as our primary mathematical tool.
- **Chapter 4** introduces the Klein-Gordon equation, its Lagrangian formulation, and the associated stress-energy-momentum tensor, deriving a local conservation law.
- **Chapter 5** synthesizes these elements to establish global conservation laws and provides a simple yet powerful example on a Schwarzschild spacetime.
- **Chapter 6** briefly extends the conservation to a broader set of field equations and solutions, using weaker assumptions to redefine the problem distributionally.
- **Chapter 7** summarizes and concludes the thesis, offering an insight into the next possible research that can be done on the topic of conservation laws.

Chapter 2

Background

The study of the Klein-Gordon equation in curved spacetime necessitates a rigorous mathematical framework drawn from differential geometry and the theory of partial differential equations.

This chapter establishes the foundational concepts and notation required for the analysis of this equation. While we present the essential definitions and theorems, detailed proofs are omitted for brevity and can be found in the cited references.

2.1 Differential geometry

In order for us to be able to describe and properly understand the properties, lemmas, propositions and theorems that take place in this thesis, we have to define spacetime in a rigorous manner. For this purpose we introduce manifolds and metrics.

A manifold generalizes the notion of surfaces and curves to higher dimensions, while a metric, (more specifically, for our case, a Lorentzian metric), encodes the geometric structure of spacetime, including distances, angles, and causal relationships.

For this work, we specifically work with smooth Lorentzian manifolds with boundary. The smoothness ensures we can apply differential operators, the boundary allows for the application of Stokes' theorem and the analysis of changing quantities, and the Lorentzian structure provides the causal framework essential for physical theories.

This mathematical setting enables us to describe spacetime geometry rigorously and derive conservation laws that hold for arbitrary spacetime configurations, as we will demonstrate in subsequent chapters.

2.1.1 From topology to calculus

For the introduction of smoothness we begin with the fundamental definitions. [Lee12; Gaj24]

Definition 1. **(Manifold)** Let $\mathscr{M}$ be a topological space. We say $\mathscr{M}$ is a **topological manifold** of dimension n if it has the following properties.

1. $\mathscr{M}$ is a Hausdorff space: for any pair of distinct points p, q belonging to $\mathscr{M}$, there exists disjoint open sets $U, V \subset \mathscr{M}$ such that $p \in U$ and $q \in V$.

2. $\mathscr{M}$ is second countable, i.e., it has a countable basis for its topology.
3. $\mathscr{M}$ is locally Euclidean of dimension n: each point has a neighborhood that is homeomorphic to an open subset of $\mathbf{R}^n$.

This definition for a topological space allows us to describe complicated structures in a well behaved and understood mathematical sense. However, it gives us no way of relating subsets of the manifold, of connecting them, therefore we require a **coordinate chart**.

Definition 2. (Coordinate chart) Let $\mathscr{M}$ be a topological manifold of dimension n. A **coordinate chart** on $\mathscr{M}$ is a pair (U,φ), where U is an open subset of $\mathscr{M}$ and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$.

We can see that the existence of a coordinate chart for any point $p \in \mathscr{M}$ arises naturally from the third property. Notice that a coordinate chart does not need to cover the whole manifold, and usually does not. This means that we require a collection of coordinate charts in order to work with $\mathscr{M}$ as a whole.

Definition 3. (Atlas) Let $\mathscr{M}$ be a topological manifold, let $(U_1,\varphi_1),...,(U_m,\varphi_m)$ be a finite family of coordinate charts on $\mathscr{M}$. If $\bigcup_{i=1}^m U_i = \mathscr{M}$ we call this collection an **atlas for $\mathscr{M}$**.

As we have homeomorphisms from the manifold to $\mathbb{R}^n$, we can relate coordinate charts via a transition maps, defined as follows.

Definition 4. (Transition map) Let $\mathscr{M}$ be a topological manifold of dimension n. If (U,φ), (V,ψ) are two coordinate charts with $U \cap V \neq \emptyset$, then the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called a **transition map** from φ to ψ.

Given two coordinate charts (U,φ), (V,ψ), we call them **smoothly compatible** if $U \cap V = \emptyset$ or if $\psi \circ \varphi^{-1}$ is a diffeomorphism. As mentioned before, we want to be able to introduce a way of doing calculus on the manifold to be able to study how it changes, so we add this structure into the atlas itself.

Definition 5. (Smooth atlas) Let $\mathscr{M}$ be a topological manifold of dimension n. An atlas $\mathscr{A}$ of $\mathscr{M}$ is called **smooth atlas** if any two charts in $\mathscr{A}$ are smoothly compatible. If the atlas $\mathscr{A}$ is not properly contained in any larger smooth atlas, we call it **maximal**. If it is maximal we also refer to it as a **smooth structure on $\mathscr{M}$**.

Given the definition for the smooth structure required to do calculus on a manifold, we define a smooth manifold as follows.

Definition 6. (Smooth Manifold) Let $\mathscr{M}$ be a topological manifold and let $\mathscr{A}$ be a maximal atlas of $\mathscr{M}$, then the pair $(\mathscr{M},\mathscr{A})$ defines a smooth manifold.

This is a space where we can introduce derivatives in the ordinary sense of continuous partial derivatives of all orders.

2.1.2 The tangent space

While coordinate charts allow us to describe positions on a manifold, they do not provide a natural framework for discussing directions, velocities, or rates of change. To define derivatives and analyze how quantities vary across the manifold, we need the concept of a tangent space $T_p\mathcal{M}$ at each point p, a vector space that approximates the manifold locally. A vector field is then a smooth assignment of a tangent vector to each point, enabling us to study directional derivatives and flows globally. Let us introduce this rigorously. We start by defining a few concepts in order.

Definition 7. (Derivation) Let $\mathcal{M}$ be a smooth manifold with or without boundary, and let $p \in \mathcal{M}$. A linear map $v_p : C^\infty(\mathcal{M}) \to \mathbb{R}$ is called a **derivation** at p if it satisfies:

$$v_p(fg) = f(p)v_p(g) + g(p)v_p(f) \qquad \forall\, f, g \in C^\infty(\mathcal{M}) \tag{2.1}$$

This definition of a **derivation** is a coordinate-free simile of the Leibniz rule for derivation. It captures the algebraic behaviour of a directional derivative at p.

Definition 8. (Tangent space) The set of all derivations of $C^\infty(\mathcal{M})$ at p, denoted by $T_p\mathcal{M}$, is a vector space called the **tangent space** to $\mathcal{M}$ at p,

Definition 9. (Tangent bundle) Let $\mathcal{M}$ be a smooth manifold with or without boundary. We define the tangent bundle of $\mathcal{M}$, denoted by $T\mathcal{M}$, to be the disjoint union of the tangent spaces at all points of $\mathcal{M}$:

$$T\mathcal{M} := \bigsqcup_{p\in\mathcal{M}} T_p\mathcal{M}\,. \tag{2.2}$$

We then have $(p, v) \in T\mathcal{M}$.

Definition 10. (Local frame) A local frame for $\mathcal{M}$ is an ordered $n-$tuple of vector fields $(E_1, ..., E_n)$, defined on an open subset of $U \subseteq \mathcal{M}$ that is linearly independent and spans the tangent bundle.

The goal of defining the tangent bundle and, with it, a local frame, is to be able to construct the property of orientation properly, as follows.

2.1.3 Orientation and Integration

Orientation is a property of smooth manifolds that allows for a choice between two inequivalent ways in which objects can be situated with respect to their surroundings. We start by describing a way of defining an orientation on tangent spaces.

Definition 11. (Tangent space orientation) If $(E_1, ..., E_n)$ is an ordered local frame for $\mathcal{M}$, we define an **orientation** as an equivalence class of $(E_1, ..., E_n)$, that is, the transition matrix B^i_j between any two ordered bases $(E_1, ..., E_n)$ and $(\tilde{E}_1, ..., \tilde{E}_n)$ in an orientation has a positive determinant.

We denote the orientation it determines as $[E_1, ..., E_n]$ and the opposite orientation as $-[E_1, ..., E_n]$.

If $T\mathcal{M}$ is oriented, then any local frame in the given orientation is said to be oriented or positively oriented.

Definition 12. **(Pointwise orientation)** Let $\mathcal{M}$ be a smooth manifold, with or without boundary. A pointwise orientation on $\mathcal{M}$ is a choice of orientation of each tangent space.

Notice that pointwise orientation does not imply that there will be a continuous orientation between different tangent spaces, we fix this with the following.

Definition 13. **(Continuous pointwise orientation)** A pointwise orientation is said to be continuous if every point of $\mathcal{M}$ is in the domain of an oriented local frame. Then, an **orientation on** $\mathcal{M}$ is a continuous pointwise orientation. We say $\mathcal{M}$ is **orientable** if it admits an orientation.

There is a consequence of these definitions that we will make use of later.

Proposition 1. *Let $\mathcal{M}$ be a smooth manifold, with or without boundary. Any nonvanishing $n-$form ω on $\mathcal{M}$ determines a unique orientation of $\mathcal{M}$ for which ω is positively oriented at each point. Conversely, if $\mathcal{M}$ is given an orientation, then there is a smooth nonvanishing $n-$form on $\mathcal{M}$ that is positively oriented at each point.*

That is, if we are able to define a nonvanishing $n-$form on our manifold, we are able to determine a unique orientation. We will later define what is called a *volume form*, which is a well defined nonvanishing $n-$form on $\mathcal{M}$.*

Definition 14. **(Outward pointing vector field)** Let $\mathcal{M}$ be an oriented smooth manifold with boundary. A vector field V along $\partial\mathcal{M}$ is **outward-pointing** if, for every $p \in \partial\mathcal{M}$:

1. $V_p \in T_p\mathcal{M}$ is not tangent to $\partial\mathcal{M}$ (i.e., $V_p \notin T_p(\partial\mathcal{M})$).
2. There exists a smooth curve $\gamma : [0,\varepsilon) \to \mathcal{M}$ with $\gamma(0) = p$ and $\gamma'(0) = V_p$ such that $\gamma(t) \notin \mathcal{M}$ for all $t \in (0,\varepsilon)$.
3. For any positively oriented ordered basis $(E_1,\dots,E_{n-1})$ of $T_p(\partial\mathcal{M})$, the ordered basis $(V_p, E_1,\dots,E_{n-1})$ is positively oriented in $T_p\mathcal{M}$.

Similarly, V is **inward-pointing** if conditions (1) and (2) hold, but $(V_p, E_1,\dots,E_{n-1})$ is negatively oriented in $T_p\mathcal{M}$.

Proposition 2. *Let $\mathcal{M}$ be an oriented smooth $n-$manifold with boundary. Then $\partial\mathcal{M}$ is orientable, and all outward-pointing vector fields along $\partial\mathcal{M}$ determine the same orientation on $\partial\mathcal{M}$.*

*With a specific vanishing issue on null hypersurfaces, which we will tackle later.

2.1.4 Metric structure: distance and causality

Now that we have a space with functions that can be differentiated and integrated, we have to introduce some non-abstract physical intuition behind this mathematical framework. Spacetime in relativity theory is fundamentally described by Lorentzian metrics, so we impose this structure to ground our mathematics in physical reality.

We start by motivating the necessity of a metric. Whenever we are thinking about space or spacetime we know intuitively that there should be a way of relating two points to each other, e.g. via a sense of distance between them. This is nicely encoded in the structure given by a metric.

Definition 15. (Metric on a Vector Space) Let V be a finite-dimensional real vector space. A *metric tensor* on V is a bilinear map $g : V \times V \to \mathbb{R}$ that is

- **Symmetric:** $g(u, v) = g(v, u)$ for all $u, v \in V$,
- **Non-degenerate:** if $g(u, v) = 0$ for all $v \in V$, then $u = 0$.

The pair (V, g) is called an *inner product space*. If g is also positive-definite ($g(u, u) > 0$ for all $u \neq 0$), it is a *Euclidean* inner product; otherwise (as in Lorentzian signature), it is *indefinite*.

So a metric tensor maps two vectors to a number. This number can be thought of as a distance. We can, furthermore, map vector fields to functions on the manifold.

Definition 16. (Metric tensor at a point) Let $\mathcal{M}$ be a smooth manifold and $p \in \mathcal{M}$. A metric tensor at p is a metric g_p on the tangent space $T_p(\mathcal{M})$.

Pointwise structure provides us with a local "measuring device" for the behaviour of the point, and, with it, we can extend the idea to a metric tensor field:.

Definition 17. (Metric Tensor Field) A metric tensor field on a smooth manifold $\mathcal{M}$ is a smooth assignment $p \mapsto g_p$ of a metric tensor to each tangent space $T_p(\mathcal{M})$.

Equivalently, let $U \subset \mathcal{M}$ be a subset of a manifold and define $\mathcal{X}(U)$ as the space of vector fields on U. We define the metric tensor field as a $(0,2)$-tensor field $g : \mathcal{X}(U) \times \mathcal{X}(U) \to C^\infty(U)$.

The metric tensor field, in the same way as our previous definition, requires certain conditions:

- g is bilinear.
- g is symmetric.
- g is non-degenerate.

One can also assign a *signature* to a metric. This is a condition that states how many of the eigenvalues of said metric are positive, and how many are negative. In the case of general relativity, we work with metrics that have one negative eigenvalue (or one positive, interchangeably). These metrics are called Lorentzian.

Definition 18. (Lorentzian Metric) Let $\mathcal{M}$ be a Manifold. We define a Lorentzian metric on $\mathcal{M}$ as one of signature $(-1, +1, ..., +1)$.

Given the proper definitions for the spaces and structures, we can define spacetime.

Definition 19. (Lorentzian manifold with boundary) A Lorentzian manifold with boundary is defined as a smooth manifold with boundary, endowed with the structure given by a Lorentzian metric.

Definition 19 describes our spacetime framework. As it is smooth, we can define a derivative in the following way.

Definition 20. (Exterior Derivative)[Gaj24, §2.3]
The exterior derivative $d : C^{\infty}(\mathcal{M}) \to \Omega^1(\mathcal{M})$ is such that, for any $X \in \mathcal{X}(\mathcal{M})$, we have

$$df(X) := X(f). \tag{2.3}$$

Let $\omega \in \Omega^n(\mathcal{M})$ and $\xi \in \Omega^s(\mathcal{M})$. We can extend the definition of the exterior derivative to the map $d : \Omega^n(\mathcal{M}) \times \Omega^s(\mathcal{M}) \to \Omega^{n+s+1}(\mathcal{M})$:

$$d(\omega \wedge \xi) := dw \wedge \xi + (-1)^n \omega \wedge d\xi. \tag{2.4}$$

Proposition 3. ***(Properties of exterior derivative)****[Lee12, §14]*
Given the prior definition, we have the following properties:

- *d is linear over $\mathbb{R}$.*
- *$d \circ d \equiv 0$.*
- *d commutes with pullbacks.*

Definition 21. (Integration map) [Wir24]
Let $U \subset \mathbb{R}^n$ be open and $\omega \in \Omega^n_c(U)$. The integral map $\int$ of compactly supported $n-$forms on U

$$\int_U : \Omega^n_c(U) \to \mathbb{R} \tag{2.5}$$

is such that, given any $\omega = f(x)dx^1 \wedge ... \wedge dx^n$, we have that the integral of ω is:

$$\int_U \omega := \int_U f(x)dx^1 \wedge ... \wedge dx^n \tag{2.6}$$

Given the definition for exterior derivatives and integration maps, we are able to introduce one of the most important theorems in the field of differential geometry.

Theorem 1. ***(Stokes' Theorem)****[Lee12]*
Let $\mathcal{M}$ be an oriented smooth $n-$manifold with boundary and let ω be a compactly supported (n-1)-form on $\mathcal{M}$. Then,

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega. \tag{2.7}$$

We later prove and work with the divergence theorem, which arises as a consequence of Stokes' theorem.

2.2 Partial Differential Equations

The study of partial differential equations (PDEs) provides the fundamental language for describing physical fields and their evolution in spacetime. While differential geometry builds the space, PDEs dictate the dynamics of, for example, particles and fields existing and acting upon it.

For our purposes, the most relevant class are hyperbolic partial differential equations, which characterize wave-like phenomena and possess a well-posed initial value formulation. This property is crucial for deterministic physical theories, as it ensures that future states are uniquely determined by initial data. The Klein-Gordon equation, which is the focus of this work, belongs to this class.

We will not go deep into the theory of PDEs here, and some familiarity with the notation is expected. We will, rather, highlight the essential concepts that allow for our analysis of conservation laws and energy identities in curved spacetime.

Definition 22. **(Hyperbolic Partial Differential Equation)** A partial differential equation is called hyperbolic if its characteristic surfaces are real and non-degenerate. More concretely, an $(n+1)$-dimensional second-order linear PDE of the form:

$$a^{\alpha\beta}(x)\partial_\alpha\partial_\beta\phi + b^\alpha(x)\partial_\alpha\phi + c(x)\phi = 0, \qquad \alpha, \beta = 0, \dots, n \tag{2.8}$$

is hyperbolic at a point x if the coefficient matrix $a^{\alpha\beta}(x)$ has one negative eigenvalue and the rest positive, when viewed as a quadratic form.

Definition 23. **(Well-Posedness)** A partial differential equation is said to be well-posed if it satisfies:

- **Existence**: Given suitable initial data, a solution exists.
- **Uniqueness**: This solution is unique.
- **Continuous Dependence**: The solution depends continuously on the initial data.

Hyperbolic equations typically exhibit well-posedness for their initial value formulation.

Remark 1. The Klein-Gordon equation $(\Box_g - \mu^2)\psi = 0$ is a hyperbolic PDE on any Lorentzian manifold $(\mathcal{M}, g)$ given global hyperbolicity. Its wave-like nature and well-posed initial value formulation make it physically meaningful, while being mathematically tractable.

Chapter 3

Divergence Theorem

In the study of physics in curved spacetime, the divergence theorem serves as a foundation for understanding conservation laws. Mathematically, it relates the integral of the divergence of a vector field over a region of spacetime to the flux of that field through the region's boundary.

This theorem is particularly powerful and useful in the context of Lorentzian manifolds, where it allows us to derive global conservation laws from local differential equations, such as the local conservation of the stress-energy-momentum tensor, which we will see in chapter 4.

For instance, when combined with spacetime symmetries obtained via Killing vector fields ($\nabla_{(\mu}X_{\nu)} = 0$), the divergence theorem enables the construction of conserved quantities such as energy and angular momentum.

The aim of this chapter is to rigorously state and prove the divergence theorem for an oriented Lorentzian manifold with boundary, setting the stage for its application to proving energy identities in subsequent chapters. We start with a few more definitions.

Definition 24. (Natural volume form ε) Let $(\mathscr{M}, g)$ be a Lorentzian manifold. We define the *natural* volume form as the $(n+1)$-form $\varepsilon = \sqrt{-\det(g)}dx^0 \wedge ... \wedge dx^n$.

As mentioned in the previous chapter, as a consequence of Proposition 1, this definition ensures that the volume form is a non-vanishing $(n+1)$-form on $\mathscr{M}$ and, therefore, determines a unique orientation, up to certain hypersurface of spacetime that we will deal with in this chapter.

Definition 25. (Divergence operator) Let $(\mathscr{M}, g)$ be a $(n+1)$-dimensional Lorentzian manifold with boundary, with volume form ε. We define the divergence operator $\mathrm{div} : \mathscr{X}(M) \to C^\infty(\mathscr{M})$ by:

$$(\mathrm{div}\, X)\varepsilon = (-1)^n \star d(X^\flat), \tag{3.1}$$

with $\star : \Omega^r(\mathscr{M}) \to \Omega^{n+1-r}(\mathscr{M})$ the Hodge dual star for an n+1 dimensional manifold, and $X^\flat := g(X, \cdot) \in \Omega(\mathscr{M})$.

Definition 26. (Boundary assumption) Let $\mathscr{M}$ be a smooth Lorentzian manifold with boundary, and let Σ_s, Σ_t and Σ_n be space-like, time-like and null hypersurfaces, respectively. We will assume our boundary $\partial\mathcal{M}$ of $\mathcal{M}$ admits a decomposition given by:

$$\partial\mathcal{M} := \Sigma_s \cup \Sigma_t \cup \Sigma_n. \tag{3.2}$$

We will sometimes drop the "-like" from the hypersurfaces for ease of writing. The motivation for this splitting arises from certain issue on null hypersurfaces: they have the property, given any vector field $\tilde{X} \in \mathscr{X}(N)$, $\tilde{g}(\tilde{X}, \tilde{X}) = 0$. Here $\tilde{g} := i^* g$ is the inclusion map of the metric from the manifold to the null hypersurface. Hence we are unable to work with the inclusion metric without it being degenerate.

To deal with this issue, we introduce what we call a *rigged vector field* ξ and a *rigging vector field* ζ [GO15]. The respective proofs for the following results can be found in the paper previously cited.

Let us consider $\zeta \in \mathscr{X}(\Sigma_n)$, α the 1-form metrically equivalent to ζ, i.e. $\alpha := g(\zeta, \cdot)$. Take $\omega = i^* \alpha$ and consider the tensors $\bar{g} = g + \alpha \otimes \alpha$ and $\tilde{g} = i^* g$.

Lemma 1. *Given a point $p \in \Sigma_n$, the following statements hold:*

1. *$\bar{g}_p$ is degenerate if and only if ζ_p is timelike and unitary for g.*

2. *$\tilde{g}_p$ is Riemannian if and only if $\zeta_p \notin T_p\Sigma_n$.*

Definition 27. **(Rigging vector field)** Let Σ_n be a null hypersurface of a Lorentzian manifold. A rigging for Σ_n is a vector field ζ, defined on some open set containing Σ_n, such that $\zeta_p \notin T_p\Sigma_n$ for each $p \in \Sigma_n$.

Definition 28. **(Rigged vector field)** The **rigged vector field** of ζ is the $\tilde{g}$-metrically equivalent vector field to the 1-form ω, denoted ξ. Hence, $\xi := \tilde{g}^{-1}(\omega, \cdot)$.

With this we have the following result.

Lemma 2. *The rigged vector field ξ is the unique vector field in Σ_n such that $g(\zeta, \xi) = 1$. Moreover, $\tilde{g}(\xi, \xi) = 1$.*

Hence, we have an equivalent definition for a normal vector field on a null hypersurface. With this notion, we can start working towards the proof of the divergence theorem.

Lemma 3. *Let $(\mathcal{M}, g)$ be an $(n+1)$-dimensional Lorentzian manifold with boundary, and $\partial\mathcal{M}$ be defined as before. Let g be the corresponding Lorentzian metric and let $\tilde{g} = i^* \bar{g}$, as defined above. We define the* ***outward-pointing*** *normal vector fields (or rigging normal in the null case) as:*

$$N = \begin{cases} n_s & \text{on } \Sigma_s, \\ n_t & \text{on } \Sigma_t, \\ \xi & \text{on } \Sigma_n, \end{cases} \tag{3.3}$$

with $n_s, n_t \in \mathscr{X}(\mathcal{M})$ and $\xi \in \mathscr{X}(\Sigma_n)$, n_s, n_t the outside pointing normal to the space- and time- hypersurfaces, and ξ the rigged vector field on null- hypersurfaces, respectively. Let ε be as in Definition 24. Moreover, let $\underline{\varepsilon} = i^*(\iota_N \varepsilon)$ and $\varepsilon_n = i^*(\iota_\zeta \varepsilon)$ be the *pullback of the volume form contracted with the normal vector field N, and the pullback of the volume form contracted with the rigging vector field ζ, respectively. Then:*

$$(-1)^n i^*(\star X^\flat) = \begin{cases} -g(X,N)\underline{\varepsilon} & N \text{ on } \Sigma_s, \\ g(X,N)\underline{\varepsilon} & N \text{ on } \Sigma_t, \\ g(X,N)\varepsilon_n & N \text{ on } \Sigma_n. \end{cases} \tag{3.4}$$

Proof. We consider the three cases separately.

Case 1: Spacelike hypersurface Σ_s

Let $N = n_s$ be the unit normal. We can decompose $X = -g(X,N)N + Y$ with $g(Y,N) = 0$. This is valid since N is normal to Σ_s and g is non-degenerate. Notice that the negative sign comes from $g(n_s, n_s) = -1$ for space hypersurfaces, given our sign convention. Then:

$$\star X^\flat = -g(X,N) \star N^\flat + \star Y^\flat,$$
$$i^*(\star X^\flat) = -g(X,N) i^*(\star N^\flat) + i^*(\star Y^\flat).$$

Since Σ_s is n-dimensional, any $(n+1)$-tuple of tangent vectors is linearly dependent [Lee12, §16], so $i^*(\star Y^\flat)$ evaluates to 0. Using $\star N^\flat = (-1)^n \iota_N \varepsilon$, we get:

$$i^*(\star X^\flat) = (-1)^{n+1} g(X,N) i^*(\iota_N \varepsilon),$$
$$(-1)^n i^*(\star X^\flat) = -g(X,N)\underline{\varepsilon},$$

where $\underline{\varepsilon} = i^*(\iota_N \varepsilon)$ is the induced volume form.

Case 2: Timelike hypersurface Σ_t

Let $N = n_t$ be the unit normal. A similar decomposition works with the opposite consideration, $g(n_t, n_t) = +1$. Then $X = g(X,N)N + Y$ and:

$$i^*(\star X^\flat) = (-1)^n g(X,N) i^*(\iota_N \varepsilon),$$
$$(-1)^n i^*(\star X^\flat) = g(X,N)\underline{\varepsilon}.$$

The sign change accounts for the timelike nature and the constraint that N is an **outwards pointing** normal.

Case 3: Null hypersurface Σ_n

Let ξ be the rigged vector field with $g(\zeta,\xi) = 1$. Decompose $X = g(X,\zeta)\xi + g(X,\xi)\zeta + Y$, where Y is tangent to Σ_n. This is valid since, for any Y tangent to Σ_n, we have $g(Y,\xi) = 0$. Then:

$$\star X^\flat = (-1)^n i_X \varepsilon = (-1)^n (g(X,\zeta)\iota_\xi \varepsilon + g(X,\xi)\iota_\zeta \varepsilon + \iota_Y \varepsilon), \tag{3.5}$$
$$(-1)^n i^*(\star X^\flat) = g(X,\zeta) i^*(\iota_\xi \varepsilon) + g(X,\xi) i^*(\iota_\zeta \varepsilon) + i^*(\iota_Y \varepsilon). \tag{3.6}$$

Since ξ, Y are tangent to Σ_n, $i^*(\iota_\xi \varepsilon) = 0$ and $i^*(i_Y \varepsilon) = 0$. Then:

$$(-1)^n i^*(\star X^\flat) = g(X,\xi)\varepsilon_n, \tag{3.7}$$

where $\varepsilon_n = i^*(\iota_\zeta \varepsilon)$.

This completes the proof for all three cases. $\square$

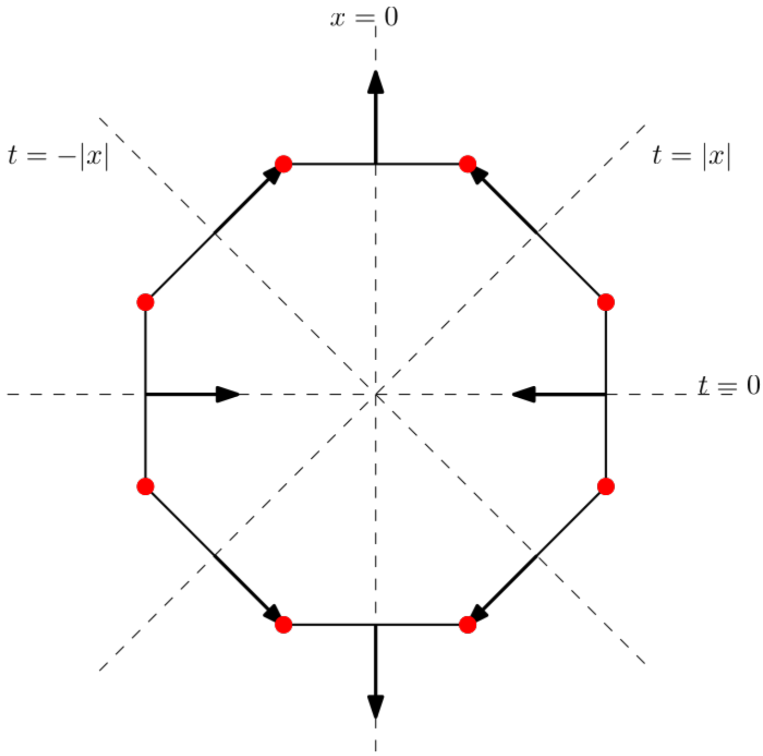

Figure 3.1: Hypersurface example with normal and rigging vectors, showcasing discontinuities at the red points. Notice the pointing of the time hypersurface null vector. As we defined N to be an outward pointing normal vector, we require a change of sign for the proof to be consistent.

There is no smoothness for N between hypersurfaces, as can be seen in the example (Figure 3.1). This highlights the need to treat the lemma and proof case-wise. As we aim to define a theorem for integration, a lemma that works almost everywhere (a.e.) is enough for a concrete proof.*

Now, given the definition of the divergence operator and the result from *Lemma* 3 we can state and prove the divergence theorem in a precise manner:

Theorem 2. ***Divergence Theorem***

Let $(\mathcal{M}, g)$ be a smooth $(n+1)$-dimensional oriented Lorentzian manifold with boundary. For any smooth vector field $X \in \mathscr{X}(\mathcal{M})$ we have:

$$\int_{\mathcal{M}} (\operatorname{div} X)\varepsilon = -\int_{\Sigma_s} g(X,N)\underline{\varepsilon} + \int_{\Sigma_t} g(X,N)\underline{\varepsilon} + \int_{\Sigma_n} g(X,N)\varepsilon_n \tag{3.8}$$

*Note that there seems to be a spin-like motion of the normal vector, this prompts the idea of extending the manifold to a complex plane and working in a different framework that may produce smoothness, but that idea is outside the scope of this work.

with N the outward pointing unit normal vector field along the boundary $\partial\mathcal{M}$, defined as before, and $\underline{\varepsilon}$, ε_n the induced volume forms on $\partial\mathcal{M}$, as stated in Lemma 3.

Proof. From the divergence theorem and Stokes' theorem (Theorem 1), we have:

$$\int_{\mathcal{M}} (\operatorname{div} X)\,\varepsilon = (-1)^n \int_{\mathcal{M}} \star d(X^\flat) = (-1)^n \int_{\partial\mathcal{M}} i^*(\star X^\flat)$$

Now, given *Lemma* 3, we split into the three cases:

$$\begin{aligned}(-1)^n \int_{\partial\mathcal{M}} i^*(\star X^\flat) =&(-1)^n \left(\int_{\Sigma_s} i^*(\star X^\flat) + \int_{\Sigma_t} i^*(\star X^\flat) + \int_{\Sigma_n} i^*(\star X^\flat)\right) \\ =& -\int_{\Sigma_s} g(X,N)\underline{\varepsilon} + \int_{\Sigma_t} g(X,N)\underline{\varepsilon} + \int_{\Sigma_n} g(X,N)\varepsilon_n\,.\end{aligned}$$

Then

$$\int_{\mathcal{M}} (\operatorname{div} X)\,\varepsilon = -\int_{\Sigma_s} g(X,N)\underline{\varepsilon} + \int_{\Sigma_t} g(X,N)\underline{\varepsilon} + \int_{\Sigma_n} g(X,N)\varepsilon_n\,.$$

□

Chapter 4

Local conservation law

In this chapter we show a local conservation law, that is, we show that the stress-energy-momentum tensor related to the Klein-Gordon equation is divergence free.

4.1 Klein-Gordon Equation

As we will be working in a general Lorentzian manifold setting, it is not enough to state the Klein-Gordon equation in a specific spacetime. We require, instead, the introduction of an operator that changes according to any metric we would want to work with.

Definition 29. (Laplace-Beltrami Operator)
Let $(\mathscr{M}, g)$ be a Lorentzian manifold. The Laplace-Beltrami operator (or d'Alembert operator in Lorentzian geometry) is defined as:

$$\Box_g := g^{\alpha\beta}\nabla_\alpha\nabla_\beta. \tag{4.1}$$

In local coordinates, this can be expressed as:

$$\Box_g = \frac{1}{\sqrt{-\det g}}\partial_\alpha\left(\sqrt{-\det g}\, g^{\alpha\beta}\partial_\beta\right). \tag{4.2}$$

Given the operator stated in Definition 29 we can now define the Klein-Gordon equation in a general setting.

Definition 30. (Klein-Gordon Equation)
Let $(\mathscr{M}, g)$ be a Lorentzian manifold. Let $\psi \in C^\infty(\mathscr{M})$ and let $\mu \in \mathbb{R}$. The Klein-Gordon equation is defined as:

$$(\Box_g - \mu^2)\psi = 0. \tag{4.3}$$

Derivation 1. [Fol08, §4.1] A direct motivation and derivation is to consider the relativistic notation for the momentum operator $p^\mu = i\partial^\mu$ and the energy-momentum relation given by $p_\mu p^\mu = m^2$. Applying this to a field ψ, we obtain from equation (4.3):

$$\begin{aligned}(p_\mu p^\mu)\psi &= \mu^2\psi\,,\\ (g^{\mu\nu}\nabla_\mu\nabla_\nu)\psi &= \mu^2\psi\,,\\ \Box_g\psi &= \mu^2\psi\,.\end{aligned}$$

Remark 2. The question of existence and uniqueness of solutions to the Klein-Gordon equation on a general Lorentzian manifold is non-trivial. For the Cauchy problem to be well-posed, certain global conditions on the spacetime are required, with **global hyperbolicity** being the standard sufficient condition. Under this assumption, which ensures the existence of a Cauchy surface, existence and uniqueness theorems can be established using energy estimates and functional analytic methods.

4.2 Action formulation

Now that we have defined the model we work with, we develop the Lagrangian field related to it, so that we can later obtain its stress-energy-momentum tensor. One of the motivations for Lagrangian fields within this thesis is the need to establish a well formulated theory for the mathematics of quantum field theory.

Definition 31. (Action functional) For a real scalar field ψ in a Lorentzian manifold $(\mathscr{M}, g)$, the action functional is defined as:

$$S[\psi, g] = \int_{\mathscr{M}} \mathscr{L}[\psi, g]\,\epsilon\,, \tag{4.4}$$

where, in our case, we use the Lagrangian density for the Klein-Gordon field, which is:

$$\mathscr{L}[\psi, g] = \frac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi + \frac{1}{2}\mu^2\psi^2. \tag{4.5}$$

Equation (4.5) is obtained from the natural Lagrangian density, whose Euler-Lagrange variation returns the Klein-Gordon equation.

Definition 32. (Stress-energy-momentum tensor) [Gaj24, §2.7] Given a solution ψ to equation (4.3), the stress-energy tensor is defined as:

$$\mathbf{T}_{\mu\nu}[\psi] = \nabla_\mu\psi\,\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\left((g^{-1})^{\alpha\beta}\nabla_\alpha\psi\,\nabla_\beta\psi + \mu^2\psi^2\right). \tag{4.6}$$

The stress-energy-momentum tensor is derived from the variation of the action corresponding to the Lagrangian field.

We will write $\mathbf{T}_{\mu\nu}[\psi]$ as $\mathbf{T}_{\mu\nu}$, while assuming it to be related to a solution of equation (4.3). This tensor satisfies the following identity.

Proposition 4. *For any solution $\psi \in C^\infty(\mathscr{M})$ of* (4.3), *the stress-energy-momentum tensor* (4.6) *obeys*

$$\nabla^\mu\mathbf{T}_{\mu\nu} = (\Box_g - \mu^2)\psi\,\nabla_\nu\psi. \tag{4.7}$$

Proof. We apply a divergence to equation (4.6). We use the Leibniz rule and we take into consideration the metric compatibility condition $\nabla g = \nabla g^{-1} = 0$:

$$\begin{aligned}\nabla^{\gamma}\mathbf{T}_{\mu\nu} &= \nabla^{\gamma}\left(\nabla_{\mu}\psi\,\nabla_{\nu}\psi\right) - \frac{1}{2}g_{\mu\nu}\left((g^{-1})^{\alpha\beta}\nabla^{\gamma}\left(\nabla_{\alpha}\psi\,\nabla_{\beta}\psi\right) + \mu^2\nabla^{\gamma}\psi^2\right)\\ &= \left(\nabla^{\gamma}\nabla_{\mu}\psi\right)\,\nabla_{\nu}\psi + \nabla_{\mu}\psi\,\left(\nabla^{\gamma}\nabla_{\nu}\psi\right) - g_{\mu\nu}\left((g^{-1})^{\alpha\beta}\left(\nabla^{\gamma}\nabla_{\alpha}\psi\right)\,\nabla_{\beta}\psi + 2\mu^2\psi\nabla^{\gamma}\psi\right)\end{aligned}$$

Let us assume $\gamma = \mu$. Since ψ is a scalar field, we will apply the fact that its covariant derivatives commute. We obtain:

$$\begin{aligned}\nabla^{\mu}\mathbf{T}_{\mu\nu} &= \left(\nabla^{\mu}\nabla_{\mu}\psi\right)\,\nabla_{\nu}\psi + \nabla_{\mu}\psi\,\left(\nabla^{\mu}\nabla_{\nu}\psi\right) - \frac{1}{2}\left(2(g^{-1})^{\alpha\beta}\left(\nabla_{\nu}\nabla_{\alpha}\psi\right)\,\nabla_{\beta}\psi + 2\mu^2\psi\nabla_{\nu}\psi\right)\\ &= (\Box_g - \mu^2)\psi\,\nabla_{\nu}\psi + \left(\nabla_{\mu}\psi\left(\nabla^{\mu}\nabla_{\nu}\psi\right) - (g^{-1})^{\alpha\beta}\left(\nabla_{\alpha}\nabla_{\nu}\psi\right)\nabla_{\beta}\psi\right).\end{aligned}$$

Exchanging $\beta = \mu$, as they are dummy indices, we finally obtain:

$$\begin{aligned}\nabla^{\mu}\mathbf{T}_{\mu\nu} &= (\Box_g - \mu^2)\psi\,\nabla_{\nu}\psi + \left(\nabla_{\mu}\psi\left(\nabla^{\mu}\nabla_{\nu}\psi\right) - \left(\nabla^{\mu}\nabla_{\nu}\psi\right)\nabla_{\mu}\psi\right)\\ &= \left(\left(\Box_g - \mu^2\right)\psi\right)\nabla_{\nu}\psi\,.\end{aligned}$$

□

Definition 33. **(Current)** Given a vector field X, we define J^X_μ to be a current via:

$$J^X_\mu = \mathbf{T}_{\mu\nu}X^\nu \tag{4.8}$$

Equation (4.8) represents the flow of a quantity associated to the vector field X through spacetime. Given $X^\nu = \partial_t$, J^X_μ represents the energy current.

Furthermore, (4.8) obeys the following:

$$\nabla^\mu J^X_\mu = (\nabla^\mu \mathbf{T}_{\mu\nu})X^\nu + \mathbf{T}_{\mu\nu}(\nabla^\mu X^\nu). \tag{4.9}$$

This follows directly from the Leibniz rule. In the next chapter we will see the relevance of having the correct stress-energy-momentum tensor and requiring it to be divergence free.

Chapter 5

Results

Given the proofs and preliminaries required to introduce a relevant physical result, we aim to introduce and interpret the results that have been obtained.

5.1 Global conservation law

Let us then apply the divergence theorem to equation (4.9), tied to any solution $\psi \in C^{\infty}(\mathscr{M})$ of equation (4.3) over a Lorentzian manifold-with-boundary $(\mathscr{M}, g)$:

$$\begin{aligned}\int_{\mathscr{M}} \nabla^{\mu} J_{\mu}^{X} \varepsilon &= \int_{\mathscr{M}} \left(\left(\nabla^{\mu} \mathbf{T}_{\mu\nu}\right) X^{\nu} + \mathbf{T}_{\mu\nu}(\nabla^{\mu} X^{\nu})\right) \varepsilon \\ &= \int_{\mathscr{M}} \left((\Box_{g} - \mu^{2})\psi \nabla_{\nu}\psi X^{\nu} + \mathbf{T}_{\mu\nu}(\nabla^{\mu} X^{\nu})\right) \varepsilon\,.\end{aligned} \tag{5.1}$$

Taking into consideration that, for (2), X is a vector field, we write $J_X := J_X^{\mu} = g^{\mu\nu} J_{\nu}^{X}$, such that we have:

$$\int_{\mathscr{M}} \nabla^{\mu} J_{\mu}^{X} \varepsilon = \int_{\mathscr{M}} \mathbf{T}_{\mu\nu}(\nabla^{\mu} X^{\nu}) \varepsilon = \int_{\partial\mathscr{M}} g(J_X, N)\underline{\varepsilon}\,. \tag{5.2}$$

5.1.1 Killing vector

Finally, we arrive at the heart of the conservation law that we have been working to derive.

Proposition 5. *Let $X \in \mathscr{X}(\mathscr{M})$ be a Killing vector, i.e. $\nabla_{\mu} X_{\nu} + \nabla_{\nu} X_{\mu} = 0$. Then equation (5.2) represents a conservation law:*

$$\int_{\partial\mathscr{M}} g(J_X^{\mu}, N)\underline{\varepsilon} = 0\,. \tag{5.3}$$

Proof. As the stress-energy-momentum tensor is symmetric, we have:

$$T_{\mu\nu}(\nabla^{\mu} X^{\nu}) = \frac{1}{2}(T_{\mu\nu}\nabla^{\mu} X^{\nu} + T_{\nu\mu}\nabla^{\nu} X^{\mu}) = \frac{1}{2} T_{\mu\nu}(\nabla^{\mu} X^{\nu} + \nabla^{\nu} X^{\mu}) = 0\,.$$

From this it follows that:

$$\int_{\mathcal{M}} \mathbf{T}_{\mu\nu}(\nabla^{\mu} X^{\nu})\varepsilon = \int_{\partial\mathcal{M}} g(J_X, N)\underline{\varepsilon} = 0\,.$$

$\square$

Furthermore, let $X \in \mathcal{X}(\mathcal{M})$ be a Killing vector and let $U_i \subset \partial\mathcal{M},\ i = 1,...,m$ with $\bigcup_{i=1}^{m} U_i = \partial\mathcal{M}$ and $U_i \cap U_j = \emptyset$ for $i \neq j$. Then:

$$\sum_{i=1}^{m} \int_{U_i} g(J_X, N)\underline{\varepsilon} = 0\,. \tag{5.4}$$

In particular, there may exist partitions where $\int_{U_i} g(J_X, N)\underline{\varepsilon} \neq 0$ for some i. This tells us that, while individual terms $\int_{U_i} g(J_X, N)\varepsilon$ may be non-zero, representing local fluxes of the conserved quantity, their sum must vanish. This showcases the essential physical content that we aimed to describe: the quantity that is associated with X can be redistributed spatially, but cannot be created nor destroyed. Hence we obtained a global conservation law for all well defined Killing vectors within any Lorentzian spacetime that possesses a well-posed field.

5.1.2 General vector fields

Given an arbitrary vector field, the energy identity becomes:

$$\int_{\mathcal{M}} \mathbf{T}_{\mu\nu}(\nabla^{\mu} X^{\nu})\varepsilon = \int_{\partial\mathcal{M}} g(J_X, N)\underline{\varepsilon}\,. \tag{5.5}$$

This can be interpreted as the work done against the stress-energy tensor by the deformation generated by X. This highlights the issues in conservation statements given general currents with no symmetries.

5.2 Schwarzschild spacetime

Let us take the Schwarzschild spacetime metric:

$$ds^2 = -(1 - \frac{r_s}{r})dt^2 + (1 - \frac{r_s}{r})^{-1}dr^2 + r^2 d\Omega^2, \tag{5.6}$$

and the vector fields $N_1 = a_1\partial_t$ and $N_2 = a_2\partial_r$, which represent normal vector fields pointing outwards to a hypersurface, with $a_1 = \sqrt{(1 - \frac{r_s}{r})^{-1}}$ and $a_2 = \sqrt{1 - \frac{r_s}{r}}$. Then $g(N_1, N_1) = -g(N_2, N_2) = -1$. We define a subset of spacetime encased by the hypersurface $\partial\mathcal{M}$, defined piecewise by:

$$\Sigma_{\tau_1} = \{(r,t) \in \mathbb{R}^3 \times \mathbb{R} | t = \tau_1\}, \tag{5.7}$$

$$\Sigma_{\tau_2} = \{(r,t) \in \mathbb{R}^3 \times \mathbb{R} | t = \tau_2, \tau_2 \in (\tau_1, \infty)\}, \tag{5.8}$$

$$\Sigma_{r_1} = \{(r,t) \in \mathbb{R}^3 \times \mathbb{R} | r = r_1, r_s < r_1\}, \tag{5.9}$$

$$\Sigma_{r_2} = \{(r,t) \in \mathbb{R}^3 \times \mathbb{R} | r = r_2, r_2 \in (r_1, \infty)\}. \tag{5.10}$$

Then, given our results, we have that, for $X \in \mathscr{X}(\mathscr{M})$ a Killing vector:

$$\int_{\partial\mathscr{M}} g(J_X, N)\underline{\varepsilon} = 0\,. \tag{5.11}$$

with $N = N_1$ or $N = N_2$ when integrating over the respective hypersurfaces, as follows:

$$\int_{\Sigma_{\tau_1}} g(J_X, N_1)\underline{\varepsilon} - \int_{\Sigma_{\tau_2}} g(J_X, N_1)\underline{\varepsilon} - \int_{\Sigma_{r_1}} g(J_X, N_2)\underline{\varepsilon} + \int_{\Sigma_{r_2}} g(J_X, N_2)\underline{\varepsilon} = 0\,. \tag{5.12}$$

The change of sign comes from the direction of the normal vector fields, as we require them to point outwards. Hence the result shows that the flux of energy stays constant, or that the difference of energy between two slices of space at different constant times differs by the flux of energy at the limits of space throughout the time that passed.

As we have the stress-energy-momentum tensor for the Klein-Gordon equation. Let us see how this conservation law expands.

Recall that:

$$T_{\mu\nu} = \nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right).$$

We rewrite:

$$J_X = (g^{-1})^{\mu\alpha}T_{\alpha\nu}X^\nu = (\nabla^\mu\psi\nabla_\nu\psi - \frac{1}{2}g^\mu_\nu\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right))X^\nu\,.$$

Take the Killing vector $X = \partial_t$, and equation (5.12) with the corresponding normals. We have:

$$\begin{aligned}
&\int_{\Sigma_{\tau_1}} g\left(\left(\nabla^\mu\psi\nabla_\nu\psi - \frac{1}{2}g^\mu_\nu\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\partial_t,\ a_1\partial_t\right)\underline{\varepsilon} \\
&-\int_{\Sigma_{\tau_2}} g\left(\left(\nabla^\mu\psi\nabla_\nu\psi - \frac{1}{2}g^\mu_\nu\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\partial_t,\ a_1\partial_t\right)\underline{\varepsilon} \\
&-\int_{\Sigma_{r_1}} g\left(\left(\nabla^\mu\psi\nabla_\nu\psi - \frac{1}{2}g^\mu_\nu\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\partial_t,\ a_2\partial_r\right)\underline{\varepsilon} \\
&+\int_{\Sigma_{r_2}} g\left(\left(\nabla^\mu\psi\nabla_\nu\psi - \frac{1}{2}g^\mu_\nu\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\partial_t,\ a_2\partial_r\right)\underline{\varepsilon} = 0\,.
\end{aligned}$$

Let us expand the value inside the integral into its coordinate form. Working in coordinates we identify t (and r) such that $J^\mu_X = T^\mu_t$ for $X = \partial_t$. Notice that $\{\partial_\mu\}$ is the coordinate basis related to $\{t, r, \phi, \theta\}$, so $g(\partial_t, \partial_r) = g_{tr}$. Then:

$$\begin{aligned}
&\int_{\Sigma_{\tau_1}} g_{\mu t}(\nabla^{\mu}\psi\partial_t\psi - \frac{1}{2}g^{\mu}_{t}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\partial_t\cdot\ a_1)\underline{\varepsilon}\\
&-\int_{\Sigma_{\tau_2}} g_{\mu t}(\nabla^{\mu}\psi\partial_t\psi - \frac{1}{2}g^{\mu}_{t}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\partial_t\cdot\ a_1)\underline{\varepsilon}\\
&-\int_{\Sigma_{r_1}} g_{\mu r}(\nabla^{\mu}\psi\partial_t\psi - \frac{1}{2}g^{\mu}_{t}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\partial_t\cdot\ a_2)\underline{\varepsilon}\\
&+\int_{\Sigma_{r_2}} g_{\mu r}(\nabla^{\mu}\psi\partial_t\psi - \frac{1}{2}g^{\mu}_{t}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\partial_t\cdot\ a_2)\underline{\varepsilon} = 0
\end{aligned}$$

Then, given that $g_{\mu t}$, $g_{\mu r}$ and any related components of the metric that are not g_{tt} or g_{rr} are 0, we have $\mu = t, r$. Furthermore, since $g^t_t = 1,\ g^r_t = 0$, we obtain:

$$\begin{aligned}
&\int_{\Sigma_{\tau_1}} g_{tt}\left(-\left(1-\frac{r_s}{r}\right)^{-1}\partial_t\psi\partial_t\psi - \frac{1}{2}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\ a_1\right)\underline{\varepsilon}\\
&-\int_{\Sigma_{\tau_2}} g_{tt}\left(-\left(1-\frac{r_s}{r}\right)^{-1}\partial_t\psi\partial_t\psi - \frac{1}{2}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\ a_1\right)\underline{\varepsilon}\\
&-\int_{\Sigma_{r_1}} g_{rr}\left(\left(1-\frac{r_s}{r}\right)\partial_r\psi\partial_t\psi\ a_2\right)\underline{\varepsilon} + \int_{\Sigma_{r_2}} g_{rr}\left(\left(1-\frac{r_s}{r}\right)\partial_r\psi\partial_t\psi\ a_2\right)\underline{\varepsilon} = 0.
\end{aligned}$$

The factors $-\left(1-\frac{r_s}{r}\right)^{-1}$ and $\left(1-\frac{r_s}{r}\right)$ come from raising indices using the inverse metric, so that tensors scale correctly. Then, $g_{tt}(-\left(1-\frac{r_s}{r}\right)^{-1}) = 1$ and $g_{rr}\left(1-\frac{r_s}{r}\right) = 1$. Replacing a_1, a_2 we obtain:

$$\begin{aligned}
&\int_{\Sigma_{\tau_1}} \left(\partial_t\psi\partial_t\psi + \frac{\left(1-\frac{r_s}{r}\right)}{2}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\sqrt{\left(1-\frac{r_s}{r}\right)^{-1}}\ \underline{\varepsilon}\\
&-\int_{\Sigma_{\tau_2}} \left(\partial_t\psi\partial_t\psi + \frac{\left(1-\frac{r_s}{r}\right)}{2}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right)\sqrt{\left(1-\frac{r_s}{r}\right)^{-1}}\ \underline{\varepsilon}\\
&-\int_{\Sigma_{r_1}} \partial_r\psi\partial_t\psi\sqrt{\left(1-\frac{r_s}{r}\right)}\ \underline{\varepsilon} + \int_{\Sigma_{r_2}} \partial_r\psi\partial_t\psi\sqrt{\left(1-\frac{r_s}{r}\right)}\ \underline{\varepsilon} = 0.
\end{aligned}$$

Finally, we have $\varepsilon = \sqrt{-\det g}\ dt\wedge dr\wedge d\theta\wedge d\phi = r^2\sin(\theta)\ dt\wedge dr\wedge d\theta\wedge d\phi$ and $\underline{\varepsilon} = \iota_N\varepsilon$. Let $f = \left(1-\frac{r_s}{r}\right)$. We obtain:

$$\begin{aligned}
&\int_{\Sigma_{\tau_1}} \left(\frac{1}{\sqrt{f}}\,\partial_t\psi\partial_t\psi + \frac{\sqrt{f}}{2}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right) r^2\sin(\theta)\ dr\ d\theta\ d\phi\\
&-\int_{\Sigma_{\tau_2}} \left(\frac{1}{\sqrt{f}}\,\partial_t\psi\partial_t\psi + \frac{\sqrt{f}}{2}\left(\left(g^{-1}\right)^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \mu^2\psi^2\right)\right) r^2\sin(\theta)\ dr\ d\theta\ d\phi\\
&-\int_{\Sigma_{r_1}} \sqrt{f}\,\partial_r\psi\partial_t\psi\ r^2\sin(\theta)\ dt\ d\theta\ d\phi + \int_{\Sigma_{r_2}} \sqrt{f}\,\partial_r\psi\partial_t\psi\ r^2\sin(\theta)\ dt\ d\theta\ d\phi = 0.
\end{aligned}$$

We can rewrite this equation as follows.

Let

$$E(\tau) := \int_{\Sigma_\tau} (\cdots)\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi,$$

and

$$\mathscr{F}(r) := \int_{\Sigma_r} \sqrt{f}\, \partial_r\psi\, \partial_t\psi\, r^2 \sin(\theta)\, dt\, d\theta\, d\phi.$$

Then:

$$E(\tau_1) - E(\tau_2) = \mathscr{F}(r_1) - \mathscr{F}(r_2) \tag{5.13}$$

The difference of the spatial energy between two given times τ_1, τ_2 is equal to the flux $\mathscr{F}(r)$ through $r = r_1$ and $r = r_2$. This expression then encodes the conservation of energy on the manifold $\mathscr{M}$.

The *flux* $\mathscr{F}(r)$, with this coordinate system, refers to the energy leaving through the boundary of a sphere of radius r.

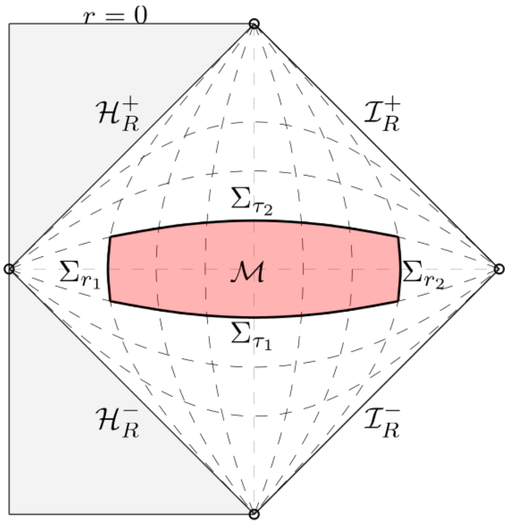

Figure 5.1: Penrose diagram of a Schwarzschild universe showing a region $\mathscr{M}$ with a boundary given by $\partial\mathscr{M}$, as defined previously.

Chapter 6

Extension

Given the obtained results, we have shown a global conservation law for the Klein-Gordon equation on any general Lorentzian spacetime. However, one can see that these results are not constrained to solutions of equation (4.3), but to divergence-free stress-energy-momentum tensors that arise from hyperbolic partial differential equations. That is, any hyperbolic PDE whose stress-energy-momentum tensor is divergence free ($\nabla^{\mu} T_{\mu\nu} = 0$) will have a conservation law, given by (5.2). Examples of such equations are:

- The wave equation $\Box\psi = f(t, \vec{x})$,
- Maxwell's equations,
- Dirac's field equation.

6.1 Weaker statement

As a continuation, we can require weaker assumptions. So far, we have assumed ψ to be smooth, i.e. $\psi \in C^{\infty}(\mathscr{M})$, and smooth solutions are, in reality, scarce. In our physical universe it is often a necessity to work with less regular functions, because of, for example, shock waves, black hole singularities or particle interactions in QFT. Physically relevant experiments and solutions often exhibit non-smooth behavior. So in order to build a framework to start working in *less than smooth* settings, we briefly build some background [Eva10, §5.2] and rewrite our conservation law:

Notation: We denote by $\alpha = (\alpha_0, ...\alpha_n)$ a multi-index such that:

$$D^{\alpha} := \frac{\partial^{\alpha_0}}{\partial x_0^{\alpha_0}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}. \tag{6.1}$$

Definition 34. (Weak solution) Let $\mathscr{M}$ be a smooth Lorentzian manifold with boundary. Suppose $u, v \in L^1_{loc}(\mathscr{M})$ and let $\alpha = (\alpha_0, ...\alpha_n)$ be a multi-index. We say v is the α^{th}-weak partial derivative of u, written $D^{\alpha} u = v$, given:

$$\int_U u D^{\alpha} \phi dx = (-1)^{|\alpha|} \int_U v \phi dx, \qquad \forall \phi \in C_c^{\infty}(\mathscr{M}). \tag{6.2}$$

This means, given u, if there exists a function v that fulfills (6.2), we say that $D^\alpha u = v$ in the *weak sense*.

Definition 35. (Sobolev spaces) The Sobolev space $W^{k,p}(U)$ consists of all locally summable functions $u : \mathscr{M} \to \mathbb{R}$ such that, for each multi-index α with, $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\mathscr{M})$.

Given these definitions, let us rewrite equation (4.3) in a weak sense. First, letting $\phi \in C_c^\infty(M)^*$, we can redefine our Klein-Gordon equation. Following (6.2):

$$\int_{\mathscr{M}} \left(\Box_g \psi - m^2\psi\right) \phi \, \varepsilon = 0\,, \tag{6.3}$$

and integrating by parts:

$$\int_{\mathscr{M}} \left(\Box_g \psi - \mu^2\psi\right) \phi \, \varepsilon = -\int_{\mathscr{M}} (g(\nabla\psi, \nabla\phi) + \mu^2 \psi\, \phi)\varepsilon = 0\,. \tag{6.4}$$

Notice, there is no boundary part as we have defined our test function ϕ to be compact. With this redefinition, we only require $\psi \in W^{1,2}_{loc}(\mathscr{M}) := H^1_{loc}(\mathscr{M})$ for it to be well defined in a distributional sense.

So we have the weaker definition of (4.3). However, we can notice an issue with our proposition for the conservation law, which requires the stress-energy-momentum tensor to be divergence-free.

Given $\psi \in H^1_{loc}(\mathscr{M})$ and $T_{\mu\nu}$ given by (4.6), we see that the divergence-free statement implies a derivative that we are unable to make sense of. Therefore we need a redefinition of our conservation law in a similar manner as we redefined the Klein-Gordon equation.

Definition 36. (Weak divergence) We say that $T_{\mu\nu}$ is *weakly divergence-free* if for every compactly supported vector field, $Y \in \mathscr{X}_c(\mathscr{M})$, we have:

$$\int_{\mathscr{M}} T_{\mu\nu} \nabla^\mu Y^\nu \, \varepsilon = 0 \tag{6.5}$$

This arises as a distributional redefinition of the divergence free condition. In return, this allows us to keep (4.6) with $\psi \in H^1_{loc}(\mathscr{M})$. We can now define the weak current.

Definition 37. (Weak current) Let $T_{\mu\nu}$ be weakly divergence-free, $X \in \mathscr{X}(\mathscr{M})$ be a vector field (not necessarily Killing), and let $f \in C_c^\infty(\mathscr{M})$. The *weak current* J^X associated with X is defined distributionally by:

$$\langle J^X, f\rangle := \int_{\mathscr{M}} T_{\mu\nu} X^\nu \nabla^\mu f \, \varepsilon\,. \tag{6.6}$$

We take its distributional divergence:

$$\langle \nabla^\mu J^X_\mu, f\rangle = \int_{\mathscr{M}} T_{\mu\nu} \nabla^\mu (X^\nu f)\, \varepsilon \tag{6.7}$$

*we generally call ϕ a *test function*

Applying the weak divergence free condition with $Y = fX$, as f is compact, gives

$$0 = \int_{\mathcal{M}} T_{\mu\nu} \nabla^{\mu}(fX^{\nu})\,\varepsilon \;= \int_{\mathcal{M}} T_{\mu\nu} \left(\nabla^{\mu} f X^{\nu} + f \nabla^{\mu} X^{\nu}\right) \varepsilon\,. \tag{6.8}$$

This yields the weak conservation law:

$$\langle \nabla^{\mu} J^{X}_{\mu}, f\rangle = \int_{\mathcal{M}} T_{\mu\nu} f \nabla^{\mu} X^{\nu}\,\varepsilon\,. \tag{6.9}$$

Then, in particular, if X is a Killing vector field (so that $\nabla_{(\mu} X_{\nu)} = 0$), the right-hand side vanishes and we obtain:

$$\langle \nabla^{\mu} J^{X}_{\mu}, f\rangle = 0 \quad \forall f \in C^{\infty}_{c}(\mathcal{M}), \tag{6.10}$$

which is the weak formulation of current conservation.

Remark 3. The choice of $H^1_{loc}(M)$ is required for physical applications: it ensures the field has locally finite energy, while also allowing the stress-energy-momentum tensor to be defined distributionally.

While one could define distributional solutions requiring only $\psi \in L^1_{\text{loc}}(M)$ via:

$$\int_{\mathcal{M}} \psi\left(\Box_g \phi - \mu^2 \phi\right) \varepsilon = 0 \quad \text{for all } \phi \in C^{\infty}_{c}(M),$$

this formulation is too weak for our purposes. As per our weak redefinition, the stress-energy tensor construction requires at least $\psi \in H^1_{\text{loc}}(M)$, making the intermediate weak formulation:

$$\int_{\mathcal{M}} \left(g(\nabla\psi, \nabla\phi) + \mu^2 \psi\phi\right) \varepsilon = 0$$

the more natural choice for extending our conservation laws to non-smooth settings.

Chapter 7

Conclusion

Throughout this thesis, it was shown that energy conservation laws in the setting of general relativity are non-trivial statements. They depend on the symmetries of the specific spacetime we work on, symmetries encoded by Killing vector fields.

We were able to precisely state how curved spacetime changes the interpretation of theorems that relate flows and hypersurfaces. They become integral evaluations of the flow of quantities that are only piecewise defined, due to Lorentzian manifolds having non-smooth transitions of normal vector fields between hypersurfaces, as illustrated in Figure 3.1.

With these considerations, the Klein-Gordon equation was used as an example of a well-posed hyperbolic partial differential equation with a divergence-free stress-energy-momentum tensor, to showcase that equations of this type obey the obtained conservation law on any smooth Lorentzian manifold, even when considering weaker assumptions on our solutions, which is of relevance when working in quantum field theory and black hole singularities.

7.1 Strategy

The obtained results required a careful introduction of the mathematical tools from differential geometry. With these we went from the definition of a manifold to the definition of a smooth Lorentzian manifold-with-boundary where we can precisely state physically relevant questions.

On this manifold we aimed to apply the divergence theorem to interpret the behavior of a flowing quantity. We had to introduce rigged and rigging vector fields to tackle null-hypersurfaces and degenerate metric definitions.

Then we introduced the stress-energy-momentum tensor related to our example of the Klein-Gordon equation. This introduced physical relevance to our question, as we can describe this geometric quantity with local conservation laws and see how, when evaluated along a Killing vector field, we obtained global conservation laws. This conservation showed, for example, that any change in our quantity after a given time is due to the influence of flows going in or out of our spacetime slice, successfully demonstrating a non-trivial conservation law on curved spacetime.

Furthermore, this law is not constrained to the Klein-Gordon equation. I holds true for any well-posed hyperbolic partial differential equation with a divergence-free stress-energy-momentum tensor. It also holds weakly via distributional definitions, as was later demonstrated, with physical relevance for H^1_{loc} solutions.

7.2 Future research directions

Given the topic of this thesis, many continuations and relevant questions can be proposed:

- What happens when we change our assumptions? For instance, one can investigate energy conservation laws in the presence of singularities, or on asymmetric spacetimes that do not admit Killing vector fields.
- How can we propose conservation laws in universes that behave dynamically? For example, we could investigate the behaviour of spacetime metrics that evolve due to moving matter.
- Given the weak formulation, how irregular can our spacetime be while still holding a physically meaningful result?
- How do conservation laws behave in a quantum setting? How would for example quantum entanglement affect them?

Conservation laws are profoundly involved in every aspect of physics and mathematics, and many follow-up questions will arise from this beautiful topic.

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