Initial value theorem for wave equations

Wave form of Maxwell's equations

Kai Uslar

Introduction

In this paper we will deal with the following concepts:

- Introduction to the wave equation
    - D'Alembert's, Poisson's and Kirchoff's formula for solving the wave equation in different dimensions.
    - The solution to the inhomogeneous wave equation from Duhamel's principle
    - Weak and strong Huygen's principle.
    - The concept of energy and causality
    - Derivation of Green's functions to prove the initial value problem.
    - Existence and Uniqueness of the solution to the wave equation.
- Maxwell's equations
    - The derivation of the wave equation for the electric field and magnetic field
    - The initial value theorem
    - The initial value theorem for Maxwell's equations

At the end of the document there will be an Appendix introducing some concepts and notation used. It will also include the diagrams for understanding Huygens principle.

Contents

Chapter 1

PDE's and the wave equation

We will start by briefly defining the concept of partial differential equation, to then proceed with the wave equation itself, the ways of solving it and describe what it tells us about the behavior of the function. [Eva10]

1.1 Partial differential equations

Definition 1. Let $u(\mathbf{x})$ be an unknown function and $k \in \mathbb{N}$, a partial differential equation can then be expressed as follows:

$$F(D^k u(\mathbf{x}), D^{k-1} u(\mathbf{x}), \ldots, Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) = 0 \qquad (\mathbf{x} \in U) \tag{1.1}$$

Where $D^k u(\mathbf{x})$ represents the set of partial derivatives:

$$D^k u(\mathbf{x}) := \left\{ D^\alpha u(\mathbf{x}) \ | \ |\alpha| = k \right\} \qquad D^\alpha u(\mathbf{x}) = \partial_{x_1}^{\alpha_1} \cdot \ldots \cdot \partial_{x_n}^{\alpha_n} u(\mathbf{x})$$

With $\alpha = (\alpha_1, \ldots, \alpha_n)$. And the function F the map:

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times U \to \mathbb{R}$$

Let $\mathbf{u}(\mathbf{x})$ be a function mapping $U \to \mathbb{R}^m$, then we have a k-th order system of partial differential equations, we will see later an example of this for the electric field and magnetic field.

1.2 Wave equation

1.2.1 Homogeneous wave equation

Definition 2. Let $u \in C^2(\mathbb{R})$ be a twice differentiable function in the set U. We define the homogeneous wave equation as follows:

$$\partial_t^2 u(t, \mathbf{x}) = \sum_{i=1}^{n} c_i \partial_{x_i}^2 u(t, \mathbf{x}) \qquad u(t, \mathbf{x}) : [0, \infty) \times U \to \mathbb{R} \tag{1.2}$$

with $U \subset \mathbb{R}^n$.

The approach to solving for u differs depending on the amount of space dimension n and requires some previous knowledge of the function itself, such as $\partial_t u(\tau,\mathbf{x}) = f(\mathbf{x})$ and $u(\tau,\mathbf{x}) = u_0(\mathbf{x})$ the so called initial values, with τ any constant time value (we will usually deal with $\tau = 0$)

The formulas to solve the homogeneous wave equation up to three dimensions are:

$$\text{For } n = 1 \rightarrow \text{d'Alembert's formula}: \qquad u(t,x) = \frac{u_0(x+c_1 t) + u_0(x-c_1 t)}{2} + \frac{1}{2}\int_{x-c_1 t}^{x+c_1 t} f(s)\,ds \tag{1.3}$$

$$\text{For } n = 2 \rightarrow \text{Poisson's formula}: \qquad u(t,x) = \frac{1}{2}\fint_{B(t,\mathbf{x})} \frac{t u_0(\mathbf{s}) + t^2 f(\mathbf{s}) + t D u_0(\mathbf{s})\cdot(\mathbf{y}-\mathbf{x})}{(t^2 - |\mathbf{s}-\mathbf{x}|^2)^{\frac{1}{2}}}\,ds \tag{1.4}$$

$$\text{For } n = 3 \rightarrow \text{Kirchhoff's formula}: \qquad u(t,\mathbf{x}) = \fint_{\partial B(t,\mathbf{s})} t f(\mathbf{x}) + u_0(\mathbf{s}) + D u_0(\mathbf{s})\cdot(\mathbf{s}-\mathbf{x})\,dS(s) \tag{1.5}$$

We will however not derive this formulas, for this you can refer to [Eva10] chapter 2.

1.2.2 Huygens principle

We slightly depart from the specific topic of wave equations to talk about the notion of causality [Str08] that arises from Huygens principle.

First of all let us introduce some subsets [Gaj22] that we will use to describe sections on spacetime, as for the rest of this document we will deal with a $\mathbb{R}^{1+3}$ spacetime, it is worth introducing the notion of lightcones C_p^+, $C_p^- \subset \mathbb{R}^{3+1}$ in Minkowski spacetime*.

C_p^+, C_p^- lightcones are defined as follows ($c = 1$ is assumed):

$$C_p^+ = p + \left\{(t,\mathbf{x}) \in \mathbb{R}^{3+1} \,\middle|\, t - |\mathbf{x}| = 0\right\}$$

$$C_p^- = p + \left\{(t,\mathbf{x}) \in \mathbb{R}^{3+1} \,\middle|\, t + |\mathbf{x}| = 0\right\}$$

Now we define the subsets I_p^+, I_p^- to be the interior of the lightcones C_p^+, C_p^-, respectively:

$$I_p^+ = p + \left\{(t,\mathbf{x}) \in \mathbb{R}^{3+1} \,\middle|\, t - |\mathbf{x}| > 0\right\}$$

$$I_p^- = \left\{(t,\mathbf{x}) \in \mathbb{R}^{3+1} \,\middle|\, t + |\mathbf{x}| < 0\right\}$$

With this subsets in mind we can explain Huygens principle in a rather simple way, we will think of two slightly different physical situations:

---

[*] Minkowski spacetime is a simple flat spacetime with Lorentzian metric $g = -dt^2 + d\mathbf{x}^2$.

(1) A collection of objects ("signals") that move with some velocity $|v| \in \mathbb{R} \leq c$ in spacetime.

(2) An electromagnetic signal in spacetime, that is, a signal with velocity $|v| = c$.

Principle 1. *We can describe Huygens principle as the boundaries of our spacetime, such that for* $V \subset \mathbb{R}^3 \times (t = \tau)$ *a sphere in space, we can define the subset of spacetime* $U \subset \mathbb{R}^{3+1}$ *to be the only subset from which every signal measured in* V *comes from. Let us use the examples to clarify what this means:*

Let $\phi(\tau, \boldsymbol{x})$ *be a signal in the subset* V*, to further ease the notion of this principle we will assume* V *to be a perfect sphere of radius* $r = R$*, this however need not be the case.*

(1) We define the ***weak Huygens principle*** *as follows: Any signal* $\phi(\tau, \boldsymbol{x}) \in V$ *is* ***at most*** *a collection of signals from the subset* $I^-_{(R,0)}$*, and for higher times it can reach at most the subset* $I^+_{(-R,0)}$

(2) We define the ***strong Huygens principle*** *as follows: Any signal* $\phi(\tau, \boldsymbol{x}) \in V$ *is* ***at most*** *a collection of signals from the subset* $I^-_{(R,0)} \setminus I^-_{(-R,0)}$*, and for higher times it can reach at most the subset* $I^+_{(-R,0)} \setminus I^+_{(R,0)}$

These can be stated a bit differently:

(1) ***Weak Huygens principle*** *tells us that any set of signals* V *can only be a consequence of causal-like signals on a time* $t < \tau$*, and can only span causal-like signals in any time* $t > \tau_f$*. That is, any signal* $\phi(t, \boldsymbol{x})$ *vanishes in the subset:*

$$\mathbb{R}^{3+1} \setminus \left(I^+_{(-R,0)} \cup I^-_{(R,0)} \right)$$

Check (3.1)

(2) ***Strong Huygens principle*** *tells us that any set of signals* V *can only be a consequence of light-like (null-like) signals on a time* $t < \tau$*, and can only span null-like signals in any time* $t > \tau$*. That is, any signal* $\phi(t, \boldsymbol{x})$ *vanishes in the subset:*

$$\left(I^+_{(R,0)} \cup I^-_{(-R,0)} \right) \cup \left(\mathbb{R}^{3+1} \setminus \left(I^+_{(-R,0)} \cup I^-_{(R,0)} \right) \right)$$

Check (3.2)

It is worth noticing that, for all electromagnetic waves $v = c$, all solutions have to be smooth. Thus as we approach the boundaries of our subset V, they have to vanish. This implies that we are not able to measure any signal in the subset $C^+_{(-R,0)} \setminus I^-_{(R,0)} \cup C^-_{(R,0)} \setminus I^+_{(-R,0)}$.

1.2.3 Inhomogeneous wave equation

Returning to the wave equation, we have seen some solutions to the homogeneous version in up to three spatial dimensions. We will use them to be able to solve the inhomogeneous case. Let us have the following:

$$\Box u(t,\mathbf{x}) = g(t,\mathbf{x}) \qquad \left(\Box = -\partial_t^2 + \sum_{i=1}^{n} c_i \partial_{x_i}^2\right) \tag{1.6}$$

$$\partial_t u(\tau,\mathbf{x}) = f(\mathbf{x}) \tag{1.7}$$

$$u(\tau,\mathbf{x}) = v(\mathbf{x}) \tag{1.8}$$

We can see already that the solutions to the homogeneous wave equation given before does not work here, as we would get an answer where $\Box u(t,\mathbf{x}) = 0$.

To solve this we will take the inhomogeneous wave equation and divide it into two separate problems: a homogeneous wave equation with given initial conditions and an inhomogeneous wave equation without initial conditions ($\partial_t w(\tau,\mathbf{x}) = 0$, $w(\tau,\mathbf{x}) = 0$):

$$\begin{aligned} \Box v(t,\mathbf{x}) &= 0 & \qquad \Box w(t,\mathbf{x}) &= g(t,\mathbf{x}) \\ \partial_t v(\tau,\mathbf{x}) &= f(\mathbf{x}) & \partial_t w(\tau,\mathbf{x}) &= 0 \\ v(\tau,\mathbf{x}) &= v(\mathbf{x}) & w(\tau,\mathbf{x}) &= 0 \end{aligned}$$

We notice that $u = v + w$ solves the initial inhomogeneous wave equation, and that we can solve v from the formulas stated above, depending on the dimensions. To solve for w we will state Duhamel's principle:

Duhamel's Principle

Principle 2. *Let $w(t,\boldsymbol{x})$ be an inhomogeneous function defined previously. Duhamel's principle states that the solution to $w(t,\boldsymbol{x})$ can be treated as a superposition of homogeneous solutions over the time interval $[0,t]$, that is, we say:* [Kad19]

$$w(x,t) = \int_0^t \phi(x,t;s)\,ds \qquad (x \in \mathbb{R}^n \times \{t \geq 0\})$$

Where $\phi(t,x)$ is a wave equation of the form

$$\begin{aligned} \Box \phi(t,\boldsymbol{x}) &= 0 \\ \partial_t \phi(\tau,\boldsymbol{x}) &= g(\tau,\boldsymbol{x}) \\ \phi(\tau,\boldsymbol{x}) &= 0 \end{aligned}$$

Example 1. For the solution of the complete inhomogeneous wave equation given some initial data, we have **(In 1 dimension)**:

$$u(t,\mathbf{x}) = \frac{v(x-t)+v(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} f(s)\,ds + \frac{1}{2}\int_0^t \phi(t,x;\tau)\,d\tau$$

$$u(t,\mathbf{x}) = \frac{v(x-t)+v(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} f(s)\,ds + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} g(\tau,\eta)\,d\tau\,d\eta \tag{1.9}$$

We can think of Duhamel's principle as the sum of all the information that conform the inhomogeneity $f(t,\mathbf{x})$ during all the time that they have existed.

1.2.4 Advanced and retarded Green's functions for inhomogeneous wave equations

Proposition 1. *Let* ψ *be a solution to* $\Box\psi(x^\mu) = -f(x^\mu)$*. Then there exists a Green's function* $G(x^\mu, x'^\mu)$ *with* $\Box G(x^\mu, x'^\mu) = -\delta(x^\mu - x'^\mu)$ *such that:*

$$\psi(x^\mu) = \int G(x^\mu, x'^\mu) f(x'^\mu) d^4 x' \tag{1.10}$$

Proof. [Wal22] So, first we want to look for a Green's function:

$$\Box_{t,x} G(t,x;t',x') = -\delta(x-x')\delta(t-t') \tag{1.11}$$

Such that given $\Box\psi(t,x) = f(t,x)$ we have:

$$\psi(t,x) = \int G(t,x;t',x') f(t',x') d^3x' dt' \tag{1.12}$$

So the approach is to solve equation (1.11) via Fourier transformations:

$$\hat{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} F(x) e^{-ikx} dx \tag{1.13}$$

The advantage of this is that we are able to take the Fourier transformation of derivatives by simply multiplying times ik:

$$\frac{\widehat{dF}}{dx}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dF}{dx}(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} ikF(x) e^{-ikx} = ik\hat{F}(k) \tag{1.14}$$

Notice we used partial integration considering that the value of our function $F(x)$ when evaluated at infinity goes to zero.
Another useful use for Fourier transformations is that it allows us to represent delta distributions applied to smooth functions that decay at infinity, that is, we can say the following:

$$\delta_{x_0}(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ikx_0} \int_{-\infty}^{\infty} dx e^{ikx} f(x)$$

This can be viewed as taking the Fourier transformation and then 'immediately' the inverse Fourier transformation to return to $\delta_{x_0}(f)$. We can then say:

$$\delta_{x_0}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ikx_0} e^{ikx} \tag{1.15}$$

Which by itself does not converge, yet holds when applied to a function.

Now we can work further with our Green function:

$$\hat{G}(\omega, k) = \frac{1}{(2\pi)^2} \int_{\infty}^{\infty} G(\omega, t) e^{+i\omega t} e^{-ik\cdot x} dt d^3 x \tag{1.16}$$

Following from eqs. (1.11), (1.14):

$$
\begin{aligned}
\Box\hat{G}(\omega,k) &= -k^2\hat{G}(\omega,k)+\omega^2\hat{G}(\omega,k)\\
(\omega^2-k^2)\hat{G}(\omega,k) &= -(\delta(t-\widehat{t')\delta(x}-x'))\\
&= \frac{1}{4\pi^2}\int_{\infty}^{\infty}\delta(t-t')\delta(x-x')e^{i\omega t}e^{-ikx}dtd^3x\\
\text{We take: } x',\ t' = 0&\\
&= -\frac{1}{4\pi^2}
\end{aligned}
$$

Then we obtain the following solution for the Fourier transform of Green's function:

$$\hat{G}(\omega,k) = -\frac{1}{4\pi^2}\frac{1}{\omega-k}\frac{1}{\omega+k} \tag{1.17}$$

Yet we notice that this is undefined for values $\omega^2 = k^2$. To solve this we infinitesimally move the function into the complex plane via adding an $i\epsilon$ with $\epsilon > 0$, and later take the limit $\epsilon \to 0$. This gives us what we will define as the *retarded* Green's function ${G_{ret}}^{\dagger}$:

$$\tilde{G}_{ret}(\omega,k) = -\frac{1}{4\pi^2}\frac{1}{\omega-k+i\epsilon}\frac{1}{\omega+k+i\epsilon} \tag{1.18}$$

And then we are allowed to take the inverse Fourier transform with respect to time:

$$\tilde{G}_{ret}(t,k) = -\frac{1}{\sqrt{2\pi}^5}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{(\omega-k+i\epsilon)(\omega+k+i\epsilon)}d\omega \tag{1.19}$$

We notice that for $t < 0$ the function is exponentially damped with no poles in the upper plane of ω, as we pushed them down by using the added imaginary part.

Therefore, we are allowed to close a contour on that half plane, which, using Cauchy's Theorem(3.5), gives us:

$$\tilde{G}_{ret}(t,k) = 0, \qquad \text{for } t < 0. \tag{1.20}$$

We can do the same for $t > 0$, However now we are in the lower plane ω and the poles at $\omega = \pm k - i\epsilon$ are present. We can apply the residue Theorem (3.6) and we obtain:

$$
\begin{aligned}
\tilde{G}_{ret}(t,k) &= \frac{1}{\sqrt{2\pi}^5}\cdot 2\pi i\left[\frac{e^{-ikt}}{2k}-\frac{e^{ikt}}{2k}\right] &\text{(1.21)}\\
&= \frac{1}{\sqrt{2\pi}^3}\frac{\sin(kt)}{k}, \qquad \text{for } t > 0. &\text{(1.22)}
\end{aligned}
$$

Next step is taking the inverse Fourier transform with respect to k, that is, returning to the original Green's function: (Remember that we are dealing with 3 space dimensions)

$$G(t,x) = \frac{1}{\sqrt{2\pi}^3}\frac{1}{\sqrt{2\pi}^3}\int_{\infty}^{\infty}\int_{\infty}^{\infty}\int_{\infty}^{\infty}\frac{\sin kt}{k}\cdot e^{ikx}d^3k$$

[†] If we instead move the function infinitesimally into the complex plane via subtracting an $i\epsilon$, then we obtain the *advanced* Green's function

$$= \frac{1}{8\pi^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\sin(kt)}{k}e^{ik|x|\cos\theta}k^2\sin\theta dk d\theta d\phi$$

$$= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\infty}\frac{\sin(kt)}{k}e^{ik|x|\cos\theta}k^2\sin\theta dk d\theta$$

$$= \frac{1}{4\pi^2}\int_0^{\infty}\frac{k^2\cdot\sin(kt)}{k}\cdot\left[\frac{e^{ik|x|}}{ik|x|}-\frac{e^{-ik|x|}}{ik|x|}\right]dk$$

$$= \frac{1}{4\pi^2|x|}\int_0^{\infty}(-i\sin(kt))\cdot\left(e^{ik|x|}-e^{-ik|x|}\right)dk$$

$$= -\frac{1}{8\pi^2|x|}\int_0^{\infty}\left(e^{ikt}-e^{-ikt}\right)\cdot\left(e^{ik|x|}-e^{-ik|x|}\right)dk$$

$$= -\frac{1}{8\pi^2|x|}\int_0^{\infty}\left(e^{ik(|x|+t)}+e^{-ik(|x|+t)}\right)-\left(e^{ik(|x|-t)}+e^{-ik(|x|-t)}\right)dk$$

$$= -\frac{1}{8\pi^2|x|}\int_{-\infty}^{\infty}\left(e^{ik(|x|+t)}-e^{ik(|x|-t)}\right)dk$$

$$= -\frac{1}{4\pi|x|}\left[\delta(t+|x|)-\delta(t-|x|)\right] \quad \text{From equation 1.15}$$

$$= \frac{\delta(t-|x|)}{4\pi|x|}, \qquad t>0. \tag{1.23}$$

We can now reinstate in our Green function the time and space initial values x', t', and the constant c that we have taken to 1[‡]:

$$G(t,x;t',x') = \frac{\delta\left(t-t'-\frac{|x-x'|}{c}\right)}{4\pi|x-x'|}, \quad t>t' \tag{1.24}$$

Therefore we get the following solution of the wave equation obtained from using Green's function:

$$\psi(t,x) = \frac{1}{4\pi}\int G(t,x;t',x')f(t,x)d^3x'dt'$$

$$\psi(t,x) = \frac{1}{4\pi}\int\frac{\delta\left(t-t'-\frac{|x-x'|}{c}\right)}{|x-x'|}f(t,x)d^3x'dt'$$

$$\psi(t,x) = \frac{1}{4\pi}\int\frac{f\left(t-\frac{|x-x'|}{c},\, x'\right)}{|x-x'|}d^3x' \tag{1.25}$$

□

[‡] Clearly they don't change the result as they can be consider as a shift of initial positions and scale, they were only excluded from the proof to ease calculations and writing.

1.2.5 Energy estimates: Existence and Uniqueness of the solution to the wave equation

Proposition 2. *Let* $u(x^\mu)$ *be a solution of the wave equation* $\Box u(x^\mu) = f(x^\mu)$*. Then* u *is unique.*

Let us define the following sets:

$U \subset \mathbb{R}^n$ bounded, open set with smooth boundary ∂U;
$U_T = U \times (0, T]$;
$\Gamma_T = \tilde{U}_T - U_T$ (Parabolic boundary of U_T), $T > 0$

Proof. [Eva10] Consider the following initial/boundary-value problem:

$$\begin{aligned} \Box u &= f && \text{in } U_T \\ \partial_t u &= g && \text{on } U \times t = 0 \\ u &= u_0 && \text{on } \Gamma_T \end{aligned}$$

Let us assume that there is another solution v such that $w = u - v$ solves the homogeneous wave equation with zero initial conditions:

$$\begin{aligned} \Box w &= 0 && \text{in } U_T \\ \partial_t w &= 0 && \text{on } U \times t = 0 \\ w &= 0 && \text{on } \Gamma_T \end{aligned}$$

We will define a time dependent function that we will call the "Energy":

$$E(t) := \frac{1}{2}\int_U (\partial_t w(t,\mathbf{x}))^2 + \left|\partial_{x_i} w(t,\mathbf{x})\right|^2 d\mathbf{x} \tag{1.26}$$

We apply a time derivative $\frac{d}{dt}$:

$$\frac{dE(t)}{dt} = \frac{1}{2}\int_U \frac{d}{dt}\left((\partial_t w)^2 + \left|\partial_{x_i} w\right|^2\right) d\mathbf{x}$$

$$\frac{dE(t)}{dt} = \frac{1}{2}\int_U 2\partial_t w \cdot \partial_t^2 w + \frac{d}{dt}\left|\partial_{x_i} w\right|^2 d\mathbf{x} \tag{1.27}$$

We will use the following property:

$$\partial_t w \Delta w = \nabla \cdot (\partial_t w \partial_{x_i} w) - \frac{d}{dt}\left(\frac{1}{2}\left|\partial_{x_i} w\right|\right)^2$$

Such that:

$$\frac{dE(t)}{dt} = \int_U \partial_t w \cdot \partial_t^2 w - \left(\partial_t w \Delta w - \nabla \cdot (\partial_t w \partial_{x_i} w)\right) d\mathbf{x} \tag{1.28}$$

$$\frac{dE(t)}{dt} = \int_U \partial_t w \cdot (\partial_t^2 w - \Delta w)\, d\mathbf{x} + \int_U \nabla \cdot (\partial_t w \partial_{x_i} w)\, d\mathbf{x} \tag{1.29}$$

Using the Divergence Theorem we have that the second integral is:

$$\int_U \nabla \cdot (\partial_t w \partial_{x_i} w)\, d\mathbf{x} = \int_{\partial U} \left(\partial_t w \partial_{x_i} w\right) \cdot \hat{n}\; dS = 0 \tag{1.30}$$

As $w = 0$ and $\partial_t w = 0$, so the integral over the boundary ∂U is 0. Hence we get:

$$\frac{dE(t)}{dt} = \int_U \partial_t w \cdot (\partial_t^2 w - \Delta w)\, d\mathbf{x} = 0 \tag{1.31}$$

This tells us that the function $E(t) = E(0) = 0$ for $0 \leq t \leq T$ due to the initial conditions/energy. Therefore $\partial_t w = 0$ and $\partial_{x_i} w = 0$ in U_T, so we can conclude that $w = u - v = 0$ and $u = v$, proving uniqueness.

□

Chapter 2

Maxwell Equations

2.1 Introduction

We have the well known differential formalism of Maxwell Equations describing the behavior of the magnetic and electric fields. To be mathematically rigorous, let us have two vector fields $\mathbf{E}(t,\mathbf{x}) : (0,\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{B}(t,\mathbf{x}) : (0,\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$ that fulfill the following equations:

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0} && (2.1)\\
\nabla\cdot\mathbf{B} &= 0 && (2.2)\\
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} && (2.3)\\
\nabla\times\mathbf{B} &= \mu_0\cdot\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} && (2.4)
\end{aligned}$$

Lemma 1. *Let* $\boldsymbol{E}$*,* $\boldsymbol{B}$ *fulfill Maxwell equations. Then they can be written in the following wave form:*

$$\begin{aligned}
\Box\boldsymbol{E} &= \nabla\left(\frac{\rho}{\epsilon_0}\right) + \frac{\partial(\mu_0 J)}{\partial t} && (2.5)\\
\Box\boldsymbol{B} &= -\mu_0\cdot\nabla\times J && (2.6)
\end{aligned}$$

Proof. The proof is relatively straight forward. We will apply some tricks to equations (2.1), (2.2), (2.3) and (2.4) to write them in a different way. First of all it is worth noticing that:

$$\Delta u = \nabla\cdot(\nabla\cdot u) - \nabla\times(\nabla\times u) \qquad (2.7)$$

Where Δ is the Laplacian, this can be easily shown. We apply a second nabla operation to all four equations as shown next:

$$\begin{aligned}
\nabla(\nabla\cdot\mathbf{E}) =& \nabla\left(\frac{\rho}{\epsilon_0}\right)\\
\nabla(\nabla\cdot\mathbf{B}) =& 0\\
\nabla\times(\nabla\times\mathbf{E}) =& \nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right)\\
\nabla\times(\nabla\times\mathbf{B}) =& \mu_0\cdot\nabla\times J+\left(\frac{1}{c^2}\frac{\partial E}{\partial t}\right)
\end{aligned}$$

Following (2.7) we subtract both equations for E and B:

$$\Delta\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) - \nabla\times(\nabla\times\mathbf{E}) = \nabla\left(\frac{\rho}{\epsilon_0}\right) - \nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right)$$

$$\Delta\mathbf{B} = \nabla(\nabla\cdot\mathbf{B}) - \nabla\times(\nabla\times\mathbf{B}) = -\mu_0\cdot\nabla\times J - \nabla\times\left(\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right)$$

We can interchange the partial derivatives on the right-hand side of the equation and use equations (2.3) and (2.4) to get:

$$\Delta\mathbf{E} = \nabla\left(\frac{\rho}{\epsilon_0}\right) - \left(-\frac{\partial(\nabla\times\mathbf{B})}{\partial t}\right) = \nabla\left(\frac{\rho}{\epsilon_0}\right) + \frac{\partial(\mu_0 J)}{\partial t} + \frac{\partial^2\mathbf{E}}{\partial t^2}$$

$$\Delta\mathbf{B} = -\mu_0\cdot\nabla\times J + \left(\frac{1}{c^2}\frac{\partial(\nabla\times\mathbf{E})}{\partial t}\right) = -\mu_0\cdot\nabla\times J + \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}$$

And now we rewrite the equations as:

$$\Box\mathbf{E} = \nabla\left(\frac{\rho}{\epsilon_0}\right) + \frac{\partial(\mu_0 J)}{\partial t} \tag{2.8}$$

$$\Box\mathbf{B} = -\mu_0\cdot\nabla\times J \tag{2.9}$$

With $\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Here we can see that this equations (2.5) and (2.6) are inhomogeneous wave equations, that is, we showed that the electric and magnetic fields can be described via wave equations that depend on the current J and charge density ρ. $\Box$

2.2 Charge-Current conservation and Maxwell equations for $t \neq 0$

2.2.1 Charge-Current conservation equation

Proposition 3. *Let $\rho(t,\boldsymbol{x})$, $J(t,\boldsymbol{x})$ be arbitrary smooth functions that fulfill Maxwell equations for any time $t = \tau$. Then they obey the following conservation equation:*

$$\frac{\partial}{\partial t}\rho(t,\boldsymbol{x}) + \nabla\cdot J(t,\boldsymbol{x}) = 0 \tag{2.10}$$

Proof. We take equation (2.4) and apply the d'Alembertian operator (□):

$$\square\left(\nabla\times\mathbf{B}-\mu_0\cdot\mathbf{J}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right)=0 \tag{2.11}$$

We expand using equations (2.5), (2.6):

$$\begin{aligned}\nabla\times\square\mathbf{B}-\mu_0\cdot\square\mathbf{J}-\frac{1}{c^2}\frac{\partial}{\partial t}\square\mathbf{E}&=0\\ -\mu_0\nabla\times\nabla\times J-\mu_0\nabla\frac{\partial}{\partial t}\rho-\mu_0\frac{\partial^2}{\partial t^2}J-\mu_0\square J&=0\end{aligned}$$

From equation 2.7 we have:

$$\begin{aligned}-\mu_0\left(\nabla(\nabla\cdot J)-\Delta J\right)-\mu_0\nabla\frac{\partial}{\partial t}\rho-\mu_0\frac{\partial^2}{\partial t^2}J-\mu_0\square J&=0\\ -\mu_0\left(\nabla(\nabla\cdot J)-\nabla\frac{\partial}{\partial t}\rho\right)+\mu_0\square J-\mu_0\square J&=0\\ -\nabla(\nabla\cdot J)-\nabla\frac{\partial}{\partial t}\rho&=0\end{aligned}$$

Which only holds if

$$\nabla\cdot J+\frac{\partial}{\partial t}\rho=0 \tag{2.12}$$

$\square$

2.2.2 Initial value theorem formulation

We will later check for existence and uniqueness of Maxwell's equations for any time τ given any initial conditions ρ and J. The approach to solve this will be to use the proved Initial value theorem for every Maxwell equation. For now we will simply state the wave form of Maxwell equations:

$$\begin{array}{cccc}\square(\nabla\cdot\mathbf{E})=\square\dfrac{\rho}{\epsilon_0} & \square(\nabla\cdot\mathbf{B})=0 & \square(\nabla\times\mathbf{E})=-\square\dfrac{\partial\mathbf{B}}{\partial t} & \square(\nabla\times\mathbf{B})=\square\left(\mu_0\cdot\mathbf{J}+\dfrac{1}{c^2}\dfrac{\partial\mathbf{E}}{\partial t}\right)\\ \partial_t\nabla\cdot\mathbf{E}_0=\partial_t\dfrac{\rho_0}{\epsilon_0} & \partial_t\nabla\cdot\mathbf{B}_0=0 & \partial_t\nabla\times\mathbf{E}_0=-\partial_t\dfrac{\partial\mathbf{B}_0}{\partial t} & \partial_t\nabla\times\mathbf{B}_0=\partial_t\left(\mu_0\cdot\mathbf{J}_0+\dfrac{1}{c^2}\dfrac{\partial\mathbf{E}_0}{\partial t}\right)\\ \nabla\cdot\mathbf{E}_0=\dfrac{\rho_0}{\epsilon_0} & \nabla\cdot\mathbf{B}_0=0 & \nabla\times\mathbf{E}_0=-\dfrac{\partial\mathbf{B}_0}{\partial t} & \nabla\times\mathbf{B}_0=\mu_0\cdot\mathbf{J}_0+\dfrac{1}{c^2}\dfrac{\partial\mathbf{E}_0}{\partial t}\end{array}$$

Theorem 1. *Let $f(t,\boldsymbol{x})$ be an arbitrary smooth function on spacetime. Let $\chi_1(\boldsymbol{x})$ and $\chi_2(\boldsymbol{x})$ be arbitrary smooth functions on space.*
Then there exist a unique smooth solution $\psi(t,\boldsymbol{x})$ of the wave equation $\square\psi(t,x)=-f(t,\boldsymbol{x})$ such that:

$$\psi(0,\boldsymbol{x})=\chi_1(\boldsymbol{x})\qquad \textit{and}\qquad \frac{\partial}{\partial t}\psi(t,\boldsymbol{x})\bigg|_{t=0}=\chi_2(\boldsymbol{x}) \tag{2.13}$$

Are satisfied initial conditions.

Proof. There are many ways of proving the initial value theorem. In a way, one was already present when we introduced Kirchhoff's formula (1.5). Yet, we will use a different approach [Wal22].

As we have proved in the previous chapter, we have a Green function that solves an inhomogeneous wave equation given certain inhomogeneity $f(t, \mathbf{x})$. We will now use it to show that we are able to describe the electromagnetic fields at all times given certain initial data.

To ease writing we will use Einstein's summation convention, that is, given two repeated indices, we implicitly assume a sum over all components, given only one (dummy index) we assume it to be a vector or covector. If non are given it is a scalar value.

We define a vector and a one form in the following ways:

$$x^{\mu} = \left(x^0, x^1, x^2, x^3\right) \qquad \text{is a spacetime vector} \tag{2.14}$$

$$\partial_{\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \qquad \text{is the spacetime generalization of } \nabla. \tag{2.15}$$

We will be considering that we are in Minkowski spacetime, that is, our spacetime is defined by the following metric in matrix form:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.16}$$

From this definitions follow that we can define the D'Alembertian $\Box$ as:

$$\Box = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}. \tag{2.17}$$

Remember that we use Einstein's notation.
Let us define ψ_1, ψ_2:

$$\Box\psi_1 = -f_1, \tag{2.18}$$
$$\Box\psi_2 = -f_2. \tag{2.19}$$

Such that we have:

$$\begin{aligned} \partial_{\mu}\left[\eta^{\mu\nu}\left(\psi_1\partial_{\nu}\psi_2 - \psi_2\partial_{\nu}\psi_1\right)\right] &= \psi_1\Box\psi_2 - \psi_2\Box\psi_1 \\ &= -\left(\psi_1 f_2 - \psi_2 f_1\right) \end{aligned}$$

Let $v^{\mu} = \left(v^0, v^1, v^2, v^3\right)$ be an arbitrary differentiable vector field in $\mathbb{R}^{1+3}$. We will build a 4 dimensional rectangular region $U \subset \mathbb{R}^{1+3}$, $U = \left\{v^{\mu} | c_0^i \leq x^i \leq c_2^i,\ c_0^i, c_1^i \in \mathbb{R},\ i = (0,1,2,3)\right\}$. And we integrate it:

$$\int_U \partial_\mu v^\mu d^4x = \int_U \left[\frac{\partial v^0}{\partial x^0} + \frac{\partial v^1}{\partial x^1} + \frac{\partial v^2}{\partial x^2} + \frac{\partial v^3}{\partial x^3} \right] dx^0 dx^1 dx^2 dx^3 \tag{2.20}$$

$$= \left[\int v^0 dx^1 dx^2 dx^3 \right]_{x^0=c_0^0}^{x^0=c_1^0} + \left[\int v^1 dx^0 dx^2 dx^3 \right]_{x^1=c_0^1}^{x^1=c_1^1} \tag{2.21}$$

$$+ \left[\int v^2 dx^0 dx^1 dx^3 \right]_{x^2=c_0^2}^{x^2=c_1^2} + \left[\int v^3 dx^0 dx^1 dx^2 \right]_{x^3=c_0^3}^{x^3=c_1^3} \tag{2.22}$$

$$\tag{2.23}$$

From these we can take v^μ to be:

$$v^\mu = \eta^{\mu\nu} \left(\psi_1 \partial_\nu \psi_2 - \psi_2 \partial_\nu \psi_1 \right) \tag{2.24}$$

Such that we can now use the same integral for eq. (2.20). We will consider $\psi_1 = \psi$ with $\Box\psi = 0$, and $\psi_2 = G_{adv}(x^\mu, x'^\mu)$, with G_{adv} given in chapter 1. Furthermore we will give some extra constraints to our subset U:

$$U = \left\{ x^\mu \in \mathbb{R}^{1+3} \mid c_0^0 = 0, c_1^0 > x'^0,\ c_0^i < x'^i - x'^0,\ c_1^i > x'^i + x'^0 \right\}$$

This way we ensure that U encloses the spacetime point x'^μ, with each face far enough that it does not intersect its past light-cone. Now we solve:

$$\int_U \partial_\mu v^\mu d^4x = \int_U \partial_\mu \eta^{\mu\nu} \left(\psi \partial_\nu G_{adv}(x^\mu, x'^\mu) - G_{adv}(x^\mu, x'^\mu) \partial_\nu \psi \right) d^4x$$

$$= - \left[\int \left(\psi \partial_0 G_{adv}(x^\mu, x'^\mu) - G_{adv}(x^\mu, x'^\mu) \partial_0 \psi \right) dx^1 dx^2 dx^3 \right]_{x^0=c_0^0}^{x^0=c_1^0}$$

$$+ \left[\int \left(\psi \partial_1 G_{adv}(x^\mu, x'^\mu) - G_{adv}(x^\mu, x'^\mu) \partial_1 \psi \right) dx^0 dx^2 dx^3 \right]_{x^1=c_0^1}^{x^1=c_1^1}$$

$$+ \left[\int \left(\psi \partial_2 G_{adv}(x^\mu, x'^\mu) - G_{adv}(x^\mu, x'^\mu) \partial_2 \psi \right) dx^0 dx^1 dx^3 \right]_{x^2=c_0^2}^{x^2=c_1^2}$$

$$+ \left[\int \left(\psi \partial_3 G_{adv}(x^\mu, x'^\mu) - G_{adv}(x^\mu, x'^\mu) \partial_3 \psi \right) dx^0 dx^1 dx^2 \right]_{x^3=c_0^3}^{x^3=c_1^3}$$

We consider that $G_{adv}(x^\mu, x'^\mu) = 0$ unless x^μ lies on the past cone of x'^μ, which we avoided with the constraint to our subset. Therefore the only face that matters is the one where $x^0 = 0$, and we obtain:

$$\int_U \partial_\mu v^\mu d^4x = -\int_{x^0=0} \left(\psi \partial_0 G_{adv}(x^\mu, x'^\mu) - G_{adv}(x^\mu, x'^\mu)\partial_0 \psi\right) dx^1 dx^2 dx^3 \tag{2.25}$$

We also have the following:[*]

$$\int_U \partial_\mu v^\mu d^4x = -\int_U \left(\psi(x^\mu)\Box G_{adv}(x^\mu, x'^\mu) - G_{adv}(x^\mu, x'^\mu)\Box\psi(x^\mu)\right) = -c\psi(x'^\mu) \tag{2.26}$$

From the definition $\Box G(x^\mu, x'^\mu) = -\delta(x^\mu - x'^\mu)$. Therefore we have:

$$\psi(x'^\mu) = -\frac{1}{c}\int_{x^0=0} \left(\psi \partial_0 G_{adv}(x^\mu, x'^\mu) - G_{adv}(x^\mu, x'^\mu)\partial_0 \psi\right) dx^1 dx^2 dx^3 \tag{2.27}$$

We can exchange x^μ with x'^μ and use $G_{adv}(x'^\mu, x^\mu) = G_{ret}(x^\mu, x'^\mu)$:

$$\psi(x^\mu) = -\frac{1}{c}\int_{x'^0=0} \left(\psi(x'^\mu) \partial'_0 G_{ret}(x^\mu, x'^\mu) - G_{ret}(x^\mu, x'^\mu)\partial'_0 \psi(x'^\mu)\right) dx'^1 dx'^2 dx'^3 \tag{2.28}$$

We do a coordinate transformation into spherical coordinates, that is, we now evaluate the integral in the sphere S' created when $t = 0$, i.e. the only part of the subset U where our Green function does not vanish:

$$\psi(x^\mu) = \frac{1}{4\pi}\int_{S'} \left[\frac{1}{r'^2}\psi(\theta', \psi') + \frac{1}{r'}\left(\frac{1}{c}\frac{\partial\psi}{\partial t}(\theta', \psi') + \mathbf{r}' \cdot \nabla\hat{\psi(\theta', \psi')}\right)\right] r'^2 \sin(\theta') d\theta' d\psi' \tag{2.29}$$

This shows us that we only require the values of ψ and its time derivative at a time $t = 0$ to be able to have the full solution ψ for all time $t > 0$. In other words, a solution ψ to the homogeneous wave equation is uniquely described by $\psi(0, \mathbf{x})$ and $\frac{\partial}{\partial t}\psi(t, \mathbf{x})\Big|_{t=0}$.

If however we have an inhomogeneous wave equation $\Box\psi(x^\mu) = -f(x^\mu)$, the solution

---

[*] We consider $\Box\psi(x^\mu) = 0$, therefore the second term is zero. For the inhomogeneous case we consider it taking into account the definition of our Green's function.

only varies by adding the Green function determined in the last chapter. We have then:

$$\psi(x^{\mu}) = \frac{1}{4\pi}\int_{S'}\left[\frac{1}{r'^2}\psi(\theta',\psi') + \frac{1}{r'}\left(\frac{1}{c}\frac{\partial\psi}{\partial t}(\theta',\psi') + \mathbf{r}'\cdot\nabla\hat{\psi}(\theta',\psi')\right)\right] r'^2\sin(\theta')d\theta' d\psi'$$

$$+\int_{|x-x'|\leq ct}\frac{f(t-t'-\frac{|\mathbf{x}-\mathbf{x}'|}{c})}{4\pi|x-x'|}d^3x' \tag{2.30}$$

So now we require also our inhomogeneity to be specified everywhere on spacetime. □

2.2.3 Initial value theorem for Maxwell's Equations

Corollary 1. *Let $\rho(t,\boldsymbol{x})$ and $\boldsymbol{J}(t,\boldsymbol{x})$ be arbitrary, smooth functions describing the charge density and current density on spacetime, respectively, that solve the charge-current conservation equation. Let the initial conditions $\boldsymbol{E}_0(\boldsymbol{x})$, $\boldsymbol{B}_0(\boldsymbol{x})$ be arbitrary smooth functions such that Maxwell's equations (2.1), (2.2), (2.3) and (2.4) are fulfilled at time ($t=0$). Then there exist a unique solution for the electric field $\boldsymbol{E}(t,\boldsymbol{x})$ and magnetic field $\boldsymbol{B}(t,\boldsymbol{x})$ with $\boldsymbol{E}(t=0,\boldsymbol{x}) = \boldsymbol{E}_0(\boldsymbol{x})$, and $\boldsymbol{B}(t=0,\boldsymbol{x}) = \boldsymbol{B}_0(\boldsymbol{x})$.*

Proof. We have proven the existence and uniqueness of a solution ψ to an homogeneous and inhomogeneous wave equation. We have also shown that we can write Maxwell's equations in wave form. This in return implies that there exist a unique smooth solution for the electric and magnetic fields. □

Corollary 2. *Let $\rho(t,\boldsymbol{x})$ and $\boldsymbol{J}(t,\boldsymbol{x})$ be arbitrary, smooth functions describing the charge density and current density on spacetime, respectively, that solve the charge-current conservation equation. Let the initial conditions $\boldsymbol{E}_0(\boldsymbol{x})$, $\boldsymbol{B}_0(\boldsymbol{x})$ be arbitrary smooth functions such that Maxwell's equations (2.1), (2.2), (2.3) and (2.4) are fulfilled at time ($t=0$). Then every Maxwell equation exists and is unique for any time $t>0$.*

Proof. As we did at the beginning of this section, we can take equations (2.1), (2.2), (2.3) and (2.4), and convert them into wave equations with certain initial data. This is allowed as $\mathbf{E}(t,\mathbf{x})$, $\mathbf{B}(t,\mathbf{x})$ are smooth solutions to (2.5) and (2.6), respectively. Then, again as follows from the initial value theorem, Maxwell's equations are smooth solutions to their respective wave equations and thus exist and are unique throughout spacetime. □

Chapter 3

Appendix

In this section we will include a bit of extra information in case the reader would like to get a deeper understanding of mentioned topics, definitions and theorems used. However the information included is quite brief, just a small overview of some concepts, in case of wanting a deeper understanding please refer to the respective books in the references.

Calculus

Definition 3. (Partial Differentiation)
Partial differentiation is defined as follows:
Let $U \subset \mathbb{R}^n$ and let $f(x_1, ..., x_n)$ be a k-th continuously differentiable function ($\in C^k$) with $f : U \to \mathbb{R}^m, \quad m \leq n$. We define the partial derivative of this function with respect to a variable x_i in the following manner:

$$\frac{\partial}{\partial x_i} f(x_1, ..., x_i, ..., x_n) = \lim_{h \to 0} \frac{f(x_1, ..., x_{i-1}, x_i + h, x_{x+1}, ..., x_n) - f(x_1, ..., x_{i-1}, x_i, x_{x+1}, ..., x_n)}{h}$$

We will mostly use the notation $\frac{\partial}{\partial x_i} f = \partial_{x_i} f$, sometimes it can be also be seen written like $\frac{\partial}{\partial x_i} f = f_{x_i}$

Definition 4. (Nabla operator ∇)
The symbol ∇ (nabla) is an operator* defined as $\nabla = \sum_{i=1}^{n} \mathbf{e}_i \partial_{x_i}$ where e_i represents a base of unit vectors. With this operator we can define the following:
Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, then

Gradient: The gradient of $\mathbf{f(x)}$ is

$$\nabla \mathbf{f(x)} = \sum_{i=1}^{3} \mathbf{e}_i \partial_{x_i} f_i(\mathbf{x}) = \begin{pmatrix} \partial_{x_1} f_1 \\ \partial_{x_2} f_2 \\ \partial_{x_3} f_3 \end{pmatrix}$$

---

[*] "A function that acts on a function", we could say that it maps elements of a set of functions to a different set of functions, e.g: Let F, G be two sets of functions and $\hat{U} : F \to G$ a map, then $\hat{U}$ can be thought of as an operator.

Divergence: The divergence of $\mathbf{f}(\mathbf{x})$ is

$$\nabla \cdot \mathbf{f(x)} = \sum_{i=1}^{3} \partial_{x_i} f_i(\mathbf{x}) = \partial_{x_1} f_1 + \partial_{x_2} f_2 + \partial_{x_3} f_3$$

Curl: The curl of $\mathbf{f}(\mathbf{x})$ is

$$\nabla \times \mathbf{f(x)} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \mathbf{e}_i \partial_{x_j} f_k(\mathbf{x}) = \begin{pmatrix} \partial_{x_2} f_3 - \partial_{x_3} f_2 \\ \partial_{x_3} f_1 - \partial_{x_1} f_3 \\ \partial_{x_1} f_2 - \partial_{x_2} f_1 \end{pmatrix}$$

We also define the "Laplacian" as $\Delta = \nabla \cdot \nabla = \sum_{i=1}^{n} \partial_{x_i}^2$, and the square operator $\square = -\partial_t^2 + \sum_{i=1}^{n} \partial_{x_i}^2$ or $\square = -\partial_t^2 + \sum_{i=1}^{n} c_i \partial_{x_i}^2$ for some constants c_i as seen in the section 1.2.2

Definition 5. (Average integral)
We define the average integrals $⨍$ for a function $f : U \to \mathbb{R}^n$ as follows:

$$⨍_{B(\mathbf{x},r)} f\,dy := \frac{1}{\alpha(n) r^n} \int_{B(\mathbf{x},r)} f\,dy \;=\; \text{average of } f \text{ over the ball } B(\mathbf{x}, r)$$

$$⨍_{\partial B(\mathbf{x},r)} f\,dS := \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(\mathbf{x},r)} f\,dS \;=\; \text{average of } f \text{ over the sphere } \partial B(\mathbf{x}, r)$$

Theorem 2. *(Divergence Theorem)*
The divergence theorem, also called Gauss's theorem is as follows:
Let $F(x_1, ..., x_n)$ be a vector field in $V \subset \mathbb{R}^m$, $F : V \to \mathbb{R}^n$, $m \geq n$

$$\iiint_V (\nabla \cdot \boldsymbol{F})\, dV = \iint_{\partial V} (\boldsymbol{F} \cdot \hat{\boldsymbol{n}}) dS \tag{3.1}$$

With $\hat{n}$ being a outwards pointing vector on the surface of the volume V (the boundary ∂V)

Theorem 3. *(Green's Theorem)*
Let C be a positively oriented real "contour". Let D be the region bounded[†] by C. Let f, $g : \mathbb{R}^2 \to \mathbb{R}$ be two continuous functions, then:

$$\oint_C \left(f\,dx + g\,dy\right) = \iint_D \left(\frac{\partial}{\partial x} g - \frac{\partial}{\partial y} f\right) dA \tag{3.2}$$

integrated counterclockwise.

[†] $I(C)$ as seen in complex analysis section

Complex analysis

Taken from [How03]
Some terminology:
We define the following:
Let $z, a \in \mathbb{C}^n$, $r \in \mathbb{R}^n$, C be a contour (defined later) then:

- $C* = \{C(t) : t \in [a, b]\}$ the image of the contour
- $I(C)$ the inside of the contour C.
- $N(a, r) = \{z : |z - a| < r\}$, is defined as an open disc around a.
- $\bar{N}(a, r) = \{z : |z - a| \leq r\}$, is defines as a closed disc around a, the closure of $N(a, r)$
- $\kappa(a, r) = \{z : |z - a| = r\} = \partial N(a, r) = \partial \bar{N}(a, r)$, is a circle. The outside boundary of the disc $N(a, r)$
- $D'(a, r) = \{z : 0 < |z - a| < r\} = N(a, r) \backslash a$, a punctured disc. Defined as a disc without the complex point a.

Definition 6. (Holomorphic functions)
Let U be an open set in $\mathbb{C}^n$ and $f : U \to \mathbb{C}^m$ a function that is complex differentiable at every point in U. Then we call f holomorphic in U. Let $U = \mathbb{C}^n$, then f is called an *entire* function

Definition 7. (Contour)
A contour is defined as a type of curve in the complex plane. To be more specific, we define a contour in $\mathbb{C}$ as a piecewise smooth, simple[‡], closed curve.

Theorem 4. *(Laurent Series)*
A generalized version of Taylor series, where we allow the existence of both negative and positive powers of $(x - a)$*:*
Let f *be holomorphic in the punctured Disc* $D'(c, R)$, $R > 0$. *Then there exist* $a_n \in \mathbb{C}^n$ *such that,* $\forall z \in D'(a, R)$*:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n \tag{3.3}$$

If $0 < r < R$*, then*

$$a_n = \frac{1}{2\pi i} \int_{\kappa(c,r)} \frac{f(w)}{(w - c)^{n+1}} dw$$

Corollary 3. *(Residue)*
If the holomorphic function $f : \mathbb{C}^n \to \mathbb{C}^n$ *has a Laurent expansion, then:*

$$\int_{\kappa(c,r)} f(z) dz = 2\pi i a_{-1} \tag{3.4}$$

We call the coefficient a_{-1} *the residue of* f *at* c*, denoted by* $res(f, c)$.

[‡] A curve that does not cross itself.

Theorem 5. *(Cauchy's Theorem)*
Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic in a subset $U \subset \mathbb{C}^n$. Then for any simply closed contour C, the contour integral is 0:

$$\int_C f(z)dz = 0 \tag{3.5}$$

Theorem 6. *(Cauchy's Integral Formula)*
Let C be a contour and $f : I(C) \cup C* \to \mathbb{C}^n$ holomorphic. Then for every point in $I(C)$:

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz$$

Theorem 7. *(Residue Theorem)*
Let C be a contour and $f : U \to V$ an holomorphic function in a domain containing $(C) \cup C*$, except for finitely many poles at $c_1, c_2, ..., c_m$ in $I(C)$. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{m} res(f, c_k) \tag{3.6}$$

Where $res(f, c_k)$ refers to the residue of f at the singularity c.

Cauchy's theorem is a case where the contour has no poles, therefore there are no residues and the integral of the function on the whole simply closed contour is 0.

3.1 Diagrams

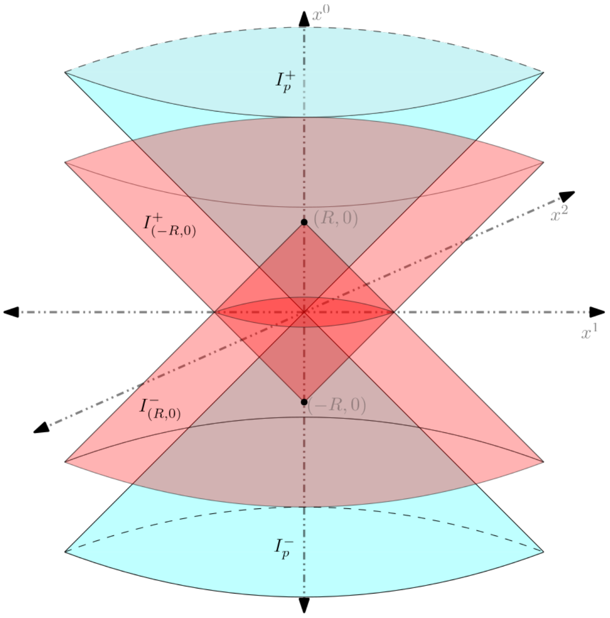

Figure 3.1: Weak Huygens principle in $\mathbb{R}^{1+2}$. The subset V would be the red circle in the center. Here, V can only be a result of the signals coming from the $I^-_{(R,0)}$.

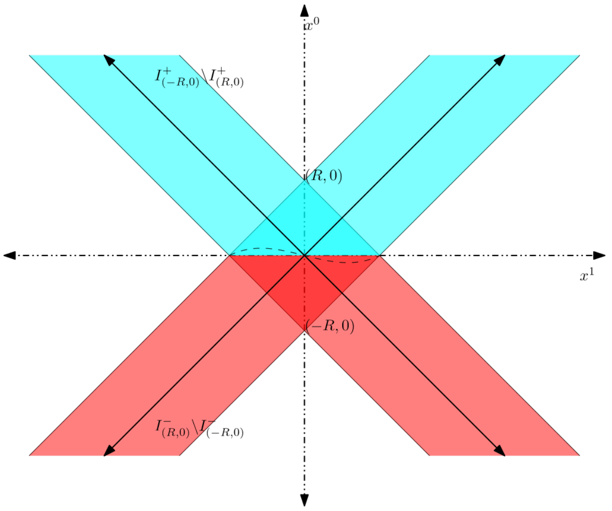

Figure 3.2: Strong Huygens principle in $\mathbb{R}^{1+1}$. The subset V is the line-spline in the center, Here every possible signal coming from our subset V follow lightlike lines.

Bibliography

[How03] John M. Howie. *Complex Analysis*. 1. Springer London, 2003. ISBN: 9781447100270.

[Str08] Walter A. Strauss. *Partial Differential Equations, an Introduction*. 1. John Wiley & Sons, 2008. ISBN: 9780470054567.

[Eva10] Lawrence C. Evans. *Partial Differentail Equations*. 2. American Mathematical Society, 2010. ISBN: 9780821849743.

[Kad19] Amir Kadivar. *Techniques for Inhomogeneous Linear PDE's*. 2019.

[Gaj22] Dejan Gajic. *Lecture notes for Nonlinear Wave Equations*. 2022.

[Wal22] Robert M. Wald. *Advanced Classical Electromagnetism*. 1. Princeton University Press, 2022. ISBN: 9780691230252.